Wasserstein contractions associated with the curvature-dimension condition

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1. Introduction

Framework

M: cpl., stoch. cpl. Riem. mfd., $\dim \geq 2$, $\partial M = \emptyset$
P_t: heat semigr. on M

<u>Goal</u>

Characterize

$$\mathrm{Ric} \geq K \ \& \ \dim M \leq N$$

in terms of behavior of couplings of heat distributions $P_t^* \mu$ $(\mu \in \mathcal{P}(M))$

lower Ricci bound on metric meas. sp.

Recent developments:

Generalization of "$\mathrm{Ric} \geq K$"

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- Geometry only based on each condition
  (even on spaces without mfd. structure)

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- Geometry only based on each condition
  (even on spaces without mfd. structure)
- Same equivalence beyond Riem. mfds

lower Ricci bound on metric meas. sp.

Why interesting?

- Geometry/Analysis on "singluar" spaces

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  - † Each cond. has its own advantage

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  for Lipschitz regularity of $P_t f$

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  ⋆ What happens when “$\mathrm{Ric} \geq K$” fails?

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    - ⇝ e.g. "Stable" sufficient cond. for Lipschitz regularity of $P_t f$
- Different viewpoints even on smooth sp.
  - ⋆ What happens when "$\mathrm{Ric} \geq K$" fails?
    - ⇝ Description of singularities in terms of P_t

lower Ricci curv. bound

For $K \in \mathbb{R}$, TFAE ([von Renesse & Sturm '05] etc.):

(i) $\mathrm{Ric} \geq K$

(ii) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$,

where $W_2(\mu_0, \mu_1) = \inf_{\pi} \|d\|_{L^2(\pi)}$

↳ coupling of μ_0 & μ_1

(iii) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

(iv) $\dfrac{1}{2}(\Delta |\nabla f|^2 - 2\langle \nabla f, \nabla \Delta f \rangle) \geq K |\nabla f|^2$

(v) Ent: K-convex w.r.t. W_2

How important?

- **(iii)(iv)** has rich applications
  in functional ineq. & differential geometry,
  e.g. quantitative Lipschitz regularization of P_t
  [Bakry & Émery etc.]

$\Rightarrow$ More applications if "$\mathrm{dim}\, M$" is involved
(e.g. Harnack inequality)

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How important?

(v) Ent: K-convex w.r.t. W_2

- **(v)** makes sense well even on singular spaces
  & stable under Gromov-Hausdorff conv.

  [Sturm '06, Lott & Villani '09]

  $\Rightarrow$ extension of **(ii)(iii)(iv)** to singular spaces

  [Ambrosio, Gigli & Savaré] etc.

Implications

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- Identification of $P_t^* \mu$ with the gradient flow of Ent in $(\mathcal{P}(M), W_2)$
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Summary of implications

$$\underset{\mathbf{Ent}}{\textbf{(v)}} \Longrightarrow \underset{\text{Wass.}}{\textbf{(ii)}} \Longleftrightarrow \underset{\text{grad.est}}{\textbf{(iii)}} \Longleftrightarrow \underset{\mathbf{\Gamma_2}}{\textbf{(iv)}}$$

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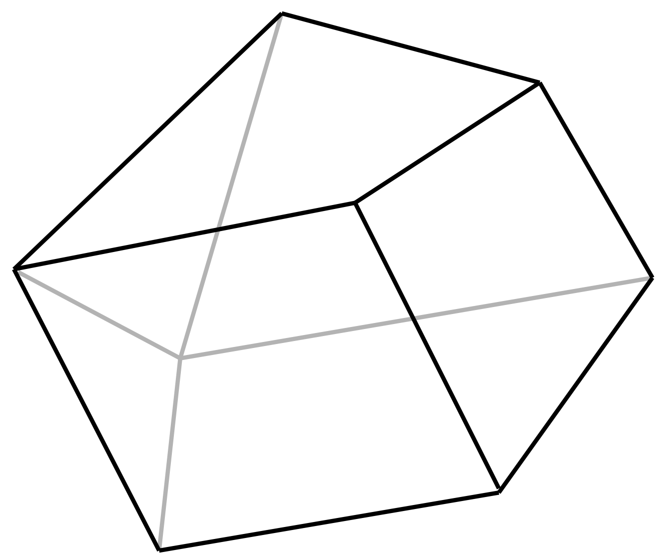
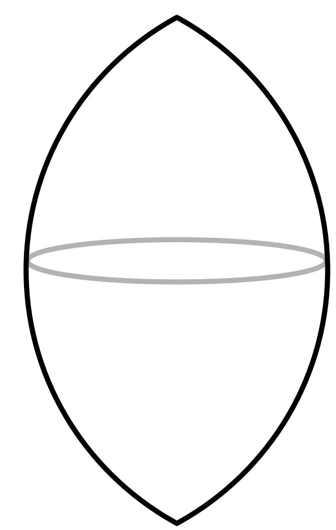
What we did for $\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$:

- Formulate a missing condition corresponding to **(ii)**
- Extension of the implication **(ii)** $\Leftrightarrow$ **(iii)** (even in an abstract setting)
- Another approach based on a coupling method

Spaces with "$\mathrm{Ric} \geq K$"

- Alexandrov sp.'s [Petrunin '11 / Gigli, K. & Ohta]
  - Boundary of a convex body (e.g. polyhedra)
  - quatient of Riem. mfd by a discrete group of isometry (orbifold)





- Wiener space ($K = 1$)
- (Finsler mfd.) [Ohta, Ohta & Sturm]
  $\leftrightsquigarrow$ Nonlinear heat equation
- (discrete Markov chains / Lévy proc.) [Maas, Erbar]

Spaces without "$\mathrm{Ric} \geq K$"

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  (e.g. Riem. surface $\leftrightarrow \sqrt{z}$)
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- Sub-Riem. mfds. ($\leftrightarrow$ hypoe. diffusion) [Juillet '09]
  - † $|\nabla P_t f| \leq C(t) P_t(|\nabla f|^q)^{1/q}$, $\lim_{t \downarrow 0} C(t) > 1$

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- Fractals [Kajino]
- Riem. mfd, $\partial M \neq \emptyset$ (F.-Y. Wang, E.P. Hsu, ...)

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Known conditions

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(i)' $\Leftrightarrow$ **(v)'**: **CD**(K, N) [Sturm '06 / Lott & Villani '09]

Theorem 1 ([K.])

For $K \in \mathbb{R}$ *and* $N \in [2, \infty]$,
(iii)' *is equivalent to the following* **(ii)'**:

(ii)' $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$

$$\leq \left(\int_s^t e^{2Kr} \xi(dr) \right)^{-1} W_2(\mu_0, \mu_1)^2 + \frac{N}{2} \xi([s,t])^2$$

where $\xi(dr) = \left(\frac{2K}{1 - e^{-2Kr}} \right)^{-1/2} dr$

The case $K = 0$

Corollary 2 ([K.])

For $N \in [2, \infty]$, TFAE:

(i)' $\mathrm{Ric} \geq 0$ & $\dim M \leq N$

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$$P_t(d(x_0, \cdot)^2)(x_1) \leq d(x_0, x_1)^2 + 2Nt$$

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$$P_t(d(x_0, \cdot)^2)(x_1) \leq d(x_0, x_1)^2 + 2Nt$$
$$\Rightarrow \Delta(d(x_0, \cdot)^2)(x_1) \leq 2N$$

(sharp Laplacian comparison)

Idea of the proof

(ii)' $\Rightarrow$ **(iii)'**: Differentiation

(iii)' $\Rightarrow$ **(ii)'**: Kantorovich duality
& analysis of the Hopf-Lax semigroup
(cf. [K. '10 / K.] when $N = \infty$)

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⇒ Extension to more general situation

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1. Introduction

2. Known results for $\mathrm{Ric} \geq K$

3. Examples

4. Curvature-dimension conditions

5. Proofs & extensions

5.1 Duality

5.2 Coupling methods

5.3 Questions

Sketch of proof: (ii)' $\Rightarrow$ (iii)'

(ii)' $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$

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$$P_t f(x) - P_s f(y) = \int (f(z) - f(w))\pi(dz dw)$$

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Take $t - s = a\, d(x, y)$ for "suitable" $a \in \mathbb{R}$:

$$\Rightarrow \frac{\text{(LHS)}}{d(x,y)} \to a \Delta P_t f(x) + |\nabla P_t f|(x) \text{ as } s \to t$$

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$$\Rightarrow \text{(RHS)} = \int \frac{f(z) - f(w)}{d(z, w)} d(z, w) \pi(dz dw)$$

$$\text{"}\le\text{"} \; P_t(|\nabla f|^2)(x)^{1/2} W_2(P_t^* \delta_x, P_s^* \delta_y) \cdots \square$$

Sketch of proof: (iii)' ⇒ (ii)'

Ingredients

- Kantorovich duality:

$$\frac{W_2(\nu,\mu)^2}{2} = \sup_f \left[\int Q_1 f \, d\mu - \int f \, d\nu \right]$$

- Hopf-Lax semigroup:

$$Q_r f(x) := \inf_{y \in M} \left[f(y) + \frac{d(x,y)^2}{2r} \right]$$

$\star$ $\partial_r Q_r f = -\frac{1}{2} |\nabla Q_r f|^2$ (Hamilton-Jacobi eq.)

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———— ∘ ———— ∘ ————

For simplicity, $\mu_0 = \delta_{x_0}$, $\mu_1 = \delta_{x_1}$

$$\frac{W_2(P_s^*\delta_{x_0}, P_t^*\delta_{x_1})^2}{2} = \sup_f \left[P_t Q_1 f(x_1) - P_s f(x_0)\right]$$

Idea: give an upper bound of $[\cdots]$ being uniform in f

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$\gamma : [0,1] \to M$: geod. joining x_0 & x_1

$\alpha : [0,1] \to [s,t]$, $\eta : [0,1] \to [0,1]$: $\nearrow$, surj.

(suitably chosen)

$$\Rightarrow P_t Q_1 f(x_1) - P_s f(x_0)$$

$$= P_{\alpha(1)} Q_1 f(\gamma(\eta(1))) - P_{\alpha(0)} Q_0 f(\gamma(\eta(0)))$$

$$= \int_0^1 \partial_r P_{\alpha(r)} Q_r f(\gamma(\eta(r))) dr$$

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$$\begin{aligned}
\partial_r P_{\alpha(r)} Q_r f(\gamma(\eta(r))) \\
&\leq \alpha'(r)\Delta P_{\alpha(r)} Q_r f(\gamma(\eta(r))) \\
&\quad - \frac{1}{2} P_{\alpha(r)}(|\nabla Q_r f|^2)(\gamma(\eta(r))) \\
&\quad + \eta'(r)|\nabla P_{\alpha(r)} Q_r f|(\gamma(\eta(r))) \\
&\leq \cdots
\end{aligned}$$

□

Remarks

- (ii) / (ii)' Wass. contr. $\underset{\text{integration}}{\overset{\text{differentiation}}{\Rightarrow \atop \Leftarrow}}$ (iii) / (iii)' gradient est.
- If an est. like (ii)' is "infinitesimally sharp", then it implies (iii)'

  $\Rightarrow$ An weaker est. than (ii)' can be equiv. to (iii)'

  $\Rightarrow$ Self-improvements in Wass. contr.'s

Extended duality

Theorem 3 ([K.])

M: Polish geod. met. sp., $P_t = e^{t\mathcal{L}}$: Feller semigr.

Then for $a, b : [0,\infty) \to (0,\infty)$, TFAE:

(A) $$W_2(P_s^*\mu_0, P_t^*\mu_1)^2 \le \left(\int_s^t \frac{\xi(dr)}{a(r)} \right)^{-1} W_2(\mu_0,\mu_1)^2 + \xi([s,t])^2$$

(B) $$|\nabla P_t f|^2 \le a(t)\left[P_t(|\nabla f|^2) + b(t)(\mathcal{L}P_t f)^2 \right]$$

where $\xi(dr) := b(r)^{-1/2} dr$,

$|\nabla f|(x)$: loc. Lip. const. of f at x

Coupling by parallel transport

$(X_0(t), X_1(t))$: coupling of BMs with different speeds

Driving noise $dB_1(t)$ of $X_1(t)$
$=$ parallel transport of $dB_0(t)$ along a geod.
& scaling

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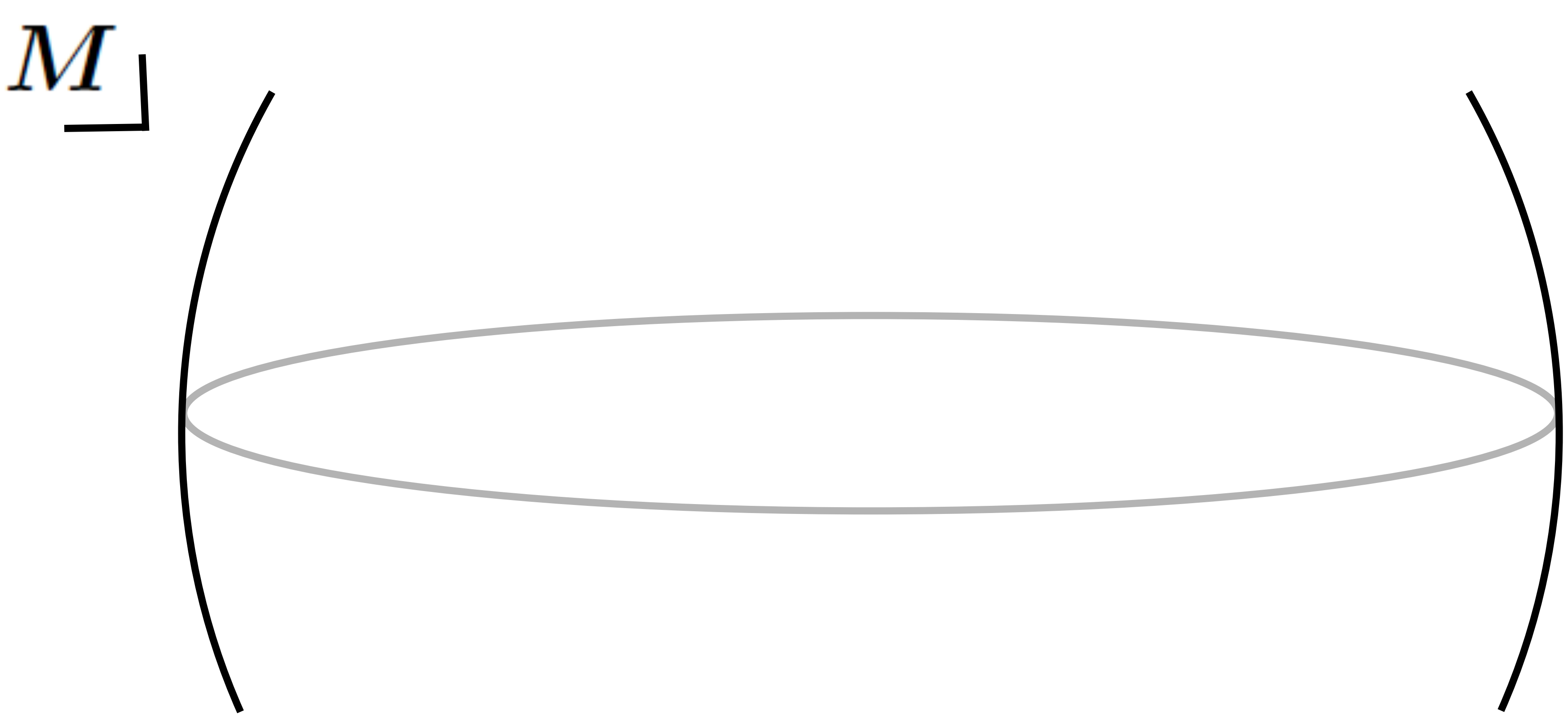

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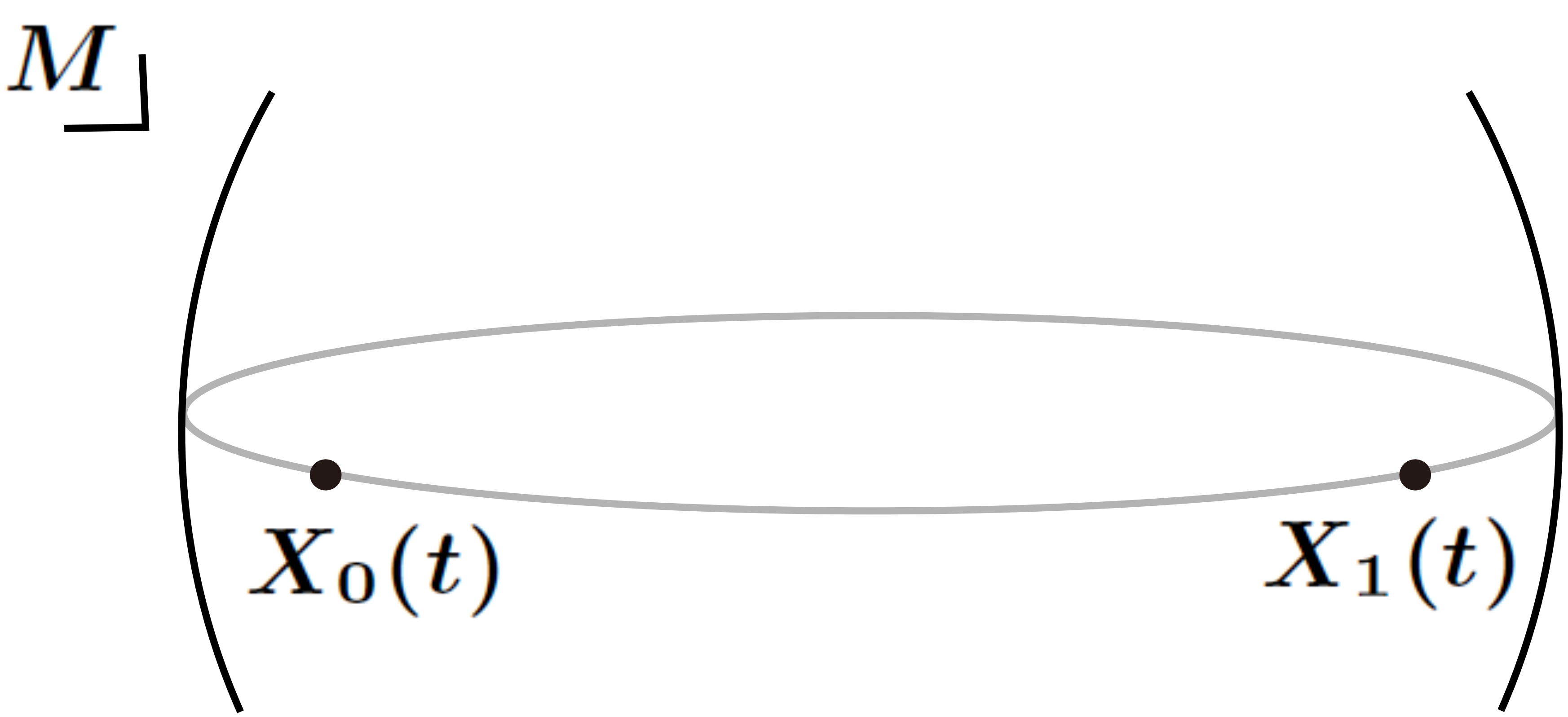

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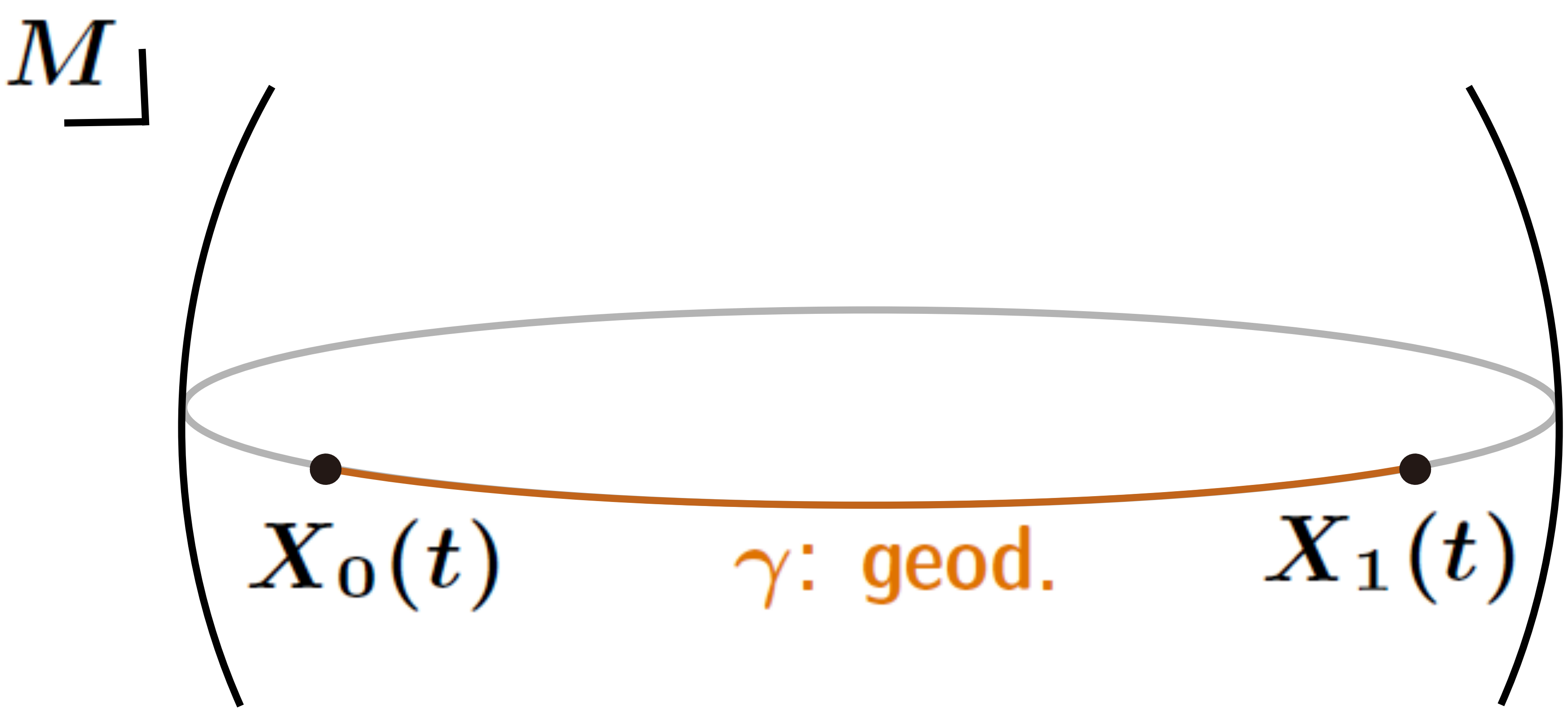

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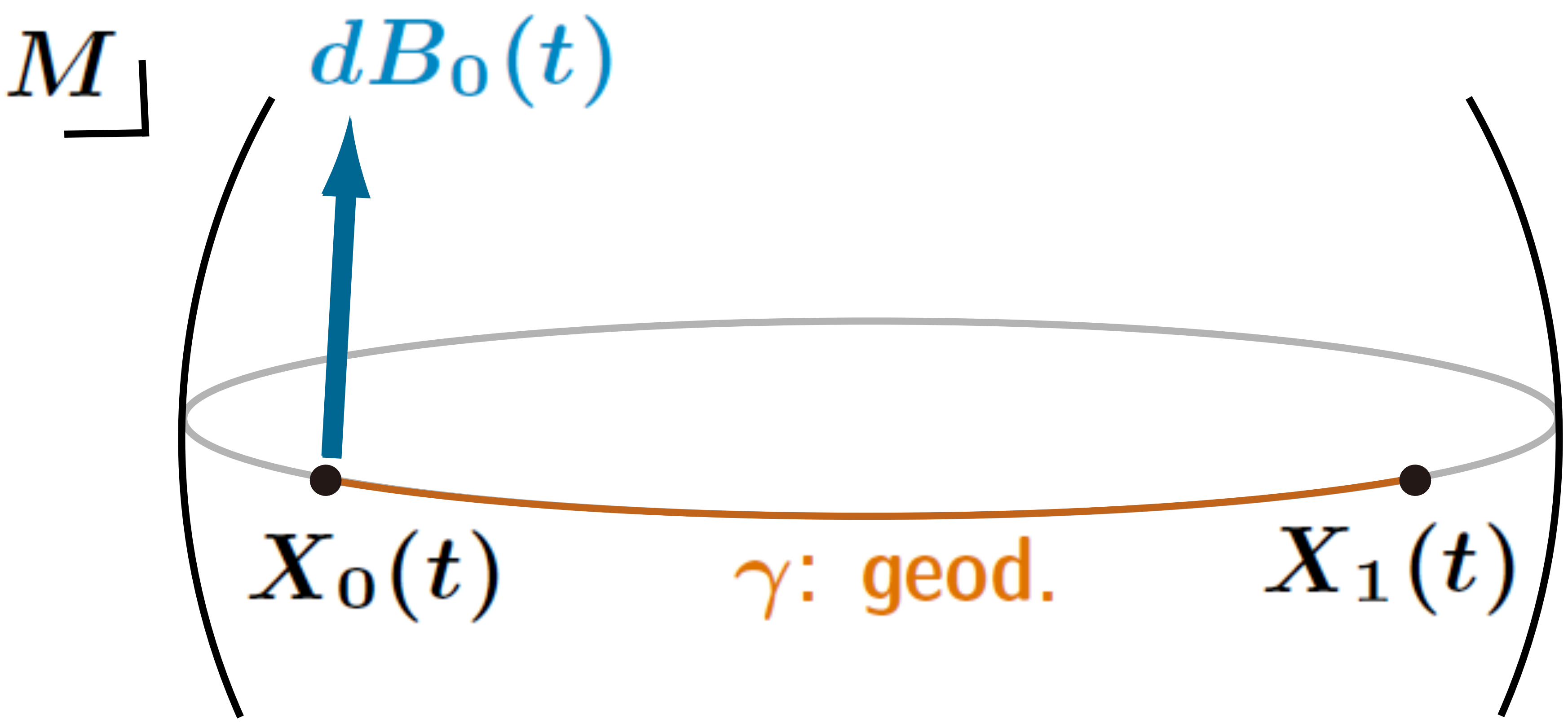

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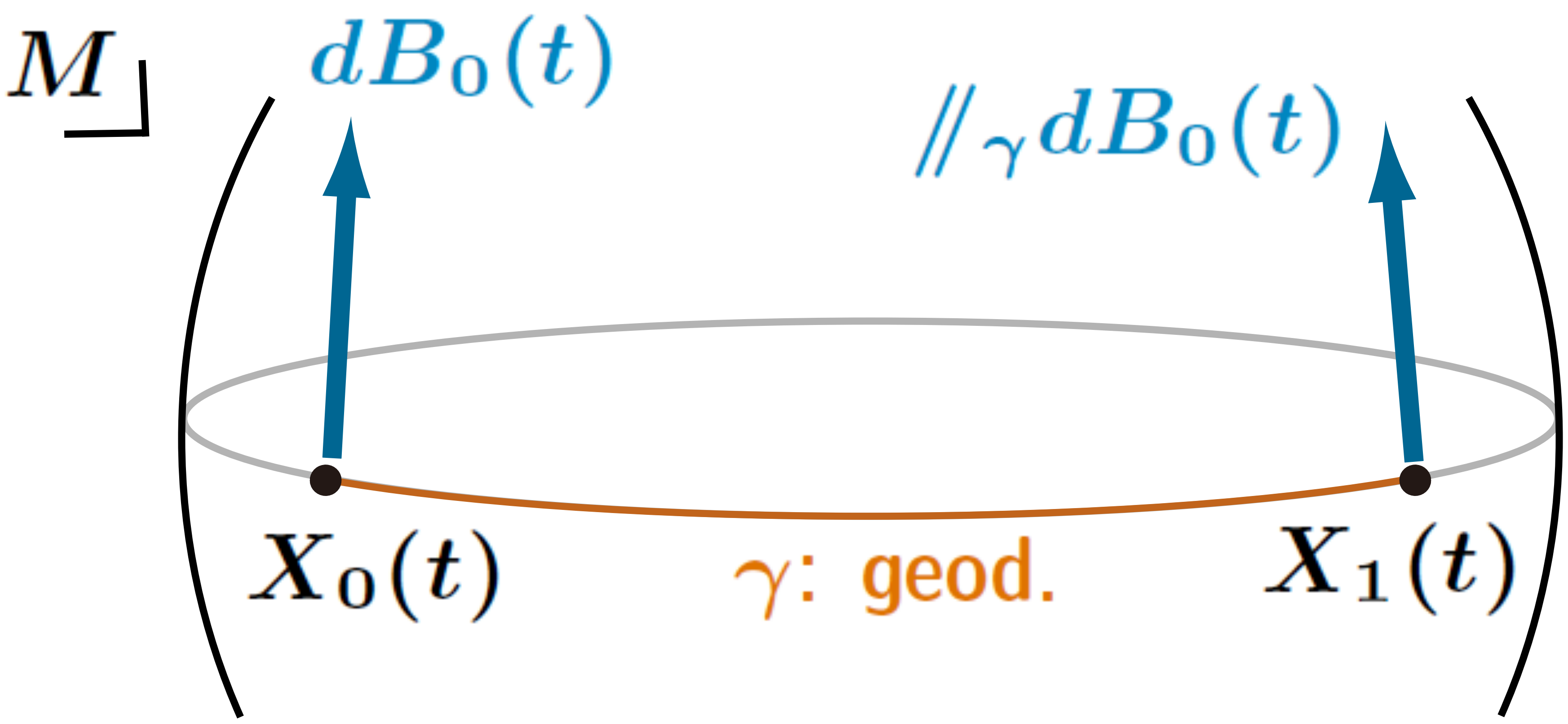

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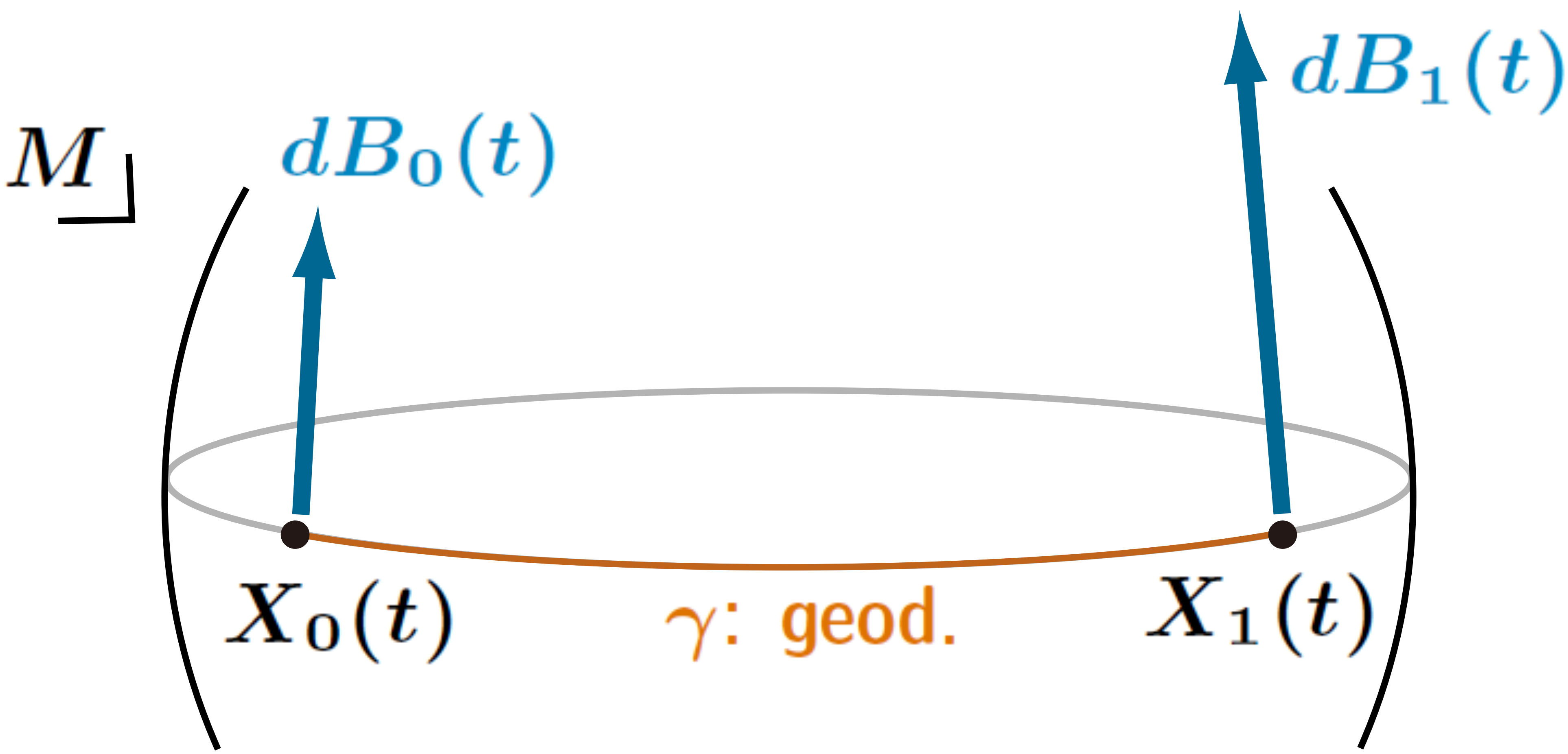

Coupling by parallel transport

$s < t$ fixed, $(\mu_r)_{r \in [0,1]}$: W_2-geod. in $\mathcal{P}(X)$
$\alpha : [0,1] \to [s,t]$, $\eta : [0,1] \to [0,1]$: $\nearrow$, surj.

$(X_r(t), X_{r'}(t))_{t \in [0,1]}$: coupling by parallel transport of
$(X(\alpha(r)t), \mathbb{P}_{\mu_{\eta(r)}})$ and $(X(\alpha(r')t), \mathbb{P}_{\mu_{\eta(r')}})$

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$$\Rightarrow W_2(P^*_{\alpha(r)}\mu_r, P^*_{\alpha(r')}\mu_{r'})^2 \le \mathbb{E}\left[d(X_r(1), X_{r'}(1))^2\right] \le \cdots$$

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$$\Rightarrow W_2(P^*_s\mu_0, P^*_t\mu_1)^2 \le \int_0^1 |\dot{P^*_{\alpha(r)}\mu_r}|^2_{W_2} dr \le \cdots,$$

where $|\dot{P^*_{\alpha(r)}\mu_r}|_{W_2} = \overline{\lim_{r'\downarrow r}} \dfrac{W_2(P^*_{\alpha(r)}\mu_r, P^*_{\alpha(r')}\mu_{r'})}{r'-r}$ $\square$

Questions

- **(v)'** $\Rightarrow$ **(ii)'** ?
  ↰ Sturm/Lott & Villani's CD$(\boldsymbol{K}, \boldsymbol{N})$
- How sharp **(ii)'** is?
  - ★ Seems to be sharp when $\boldsymbol{K} = \mathbf{0}$
    (Laplacian comparison)
- Connection with the monotonicity of normalized $\mathcal{L}$-transp. cost under backward Ricci flow?
  [cf. Topping '09, K.-Philipowski '11]