Stochastic Analysis and Applications 2012

Exponential convergence of Markovian semigroups and their spectra on L^p-spaces

(joint work with Ichiro Shigekawa)

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0. Introduction

$(M, \mathscr{B}, m)$: a probability space,

T_t: a Markovian semigroup on $L^2(m)$

i.e. $0 \le T_t f \le 1$ for $f \in L^2(m)$ and $0 \le f \le 1$.

We assume that T_t is strong continuous, $T_t \mathbf{1} = \mathbf{1}$,

T_t^* is also Markovian and $T_t^* \mathbf{1} = \mathbf{1}$.

Then, $\{T_t\}$ can be extended (or restricted)

to the Markovian semigroup on $L^p(m)$ for $p \in [1, \infty]$,

and the extension (or the restriction) of $\{T_t\}$

is strong continuous and contractive for $p \in [1, \infty)$.

Let $\langle f \rangle := \int_M f dm$ for $f \in L^1(m)$.

We are interested in the index:

$$\gamma_{p\to q} := -\limsup_{t\to\infty} \frac{1}{t} \log \|T_t - m\|_{p\to q}$$

where m means the linear operator $f \mapsto \langle f \rangle \mathbf{1}$ on $L^p(m)$ and $\|\cdot\|_{p\to q}$ is the operator norm from $L^p(m)$ to $L^q(m)$. In the case that T_t is ergodic, $\gamma_{p\to q}$ the exponential rate of the convergence.

The index $\gamma_{p\to p}$ is related to the spectra of T_t as follows:

$$\mathrm{Rad}(T_t^{(p)} - m) = e^{-\gamma_{p\to p} t}, \quad t \in [0,\infty),$$

where $\mathrm{Rad}(A)$ is the radius of spectra of A and $T_t^{(p)}$ means the linear operator T_t on $L^p(m)$.

Let $\mathfrak{A}_p$ be the generator of $\{T_t^{(p)}\}$.
If $\{T_t^{(p)}\}$ is an analytic semigroup, then

$$e^{t\sigma(\mathfrak{A}_p)\setminus\{0\}} = \sigma(T_t^{(p)} - m) \setminus \{0\}, \quad t \in [0,\infty),$$
$$\sup\{\mathrm{Re}\lambda; \lambda \in \sigma(\mathfrak{A}_p) \setminus \{0\}\} = \lim_{t\to\infty} \frac{1}{t} \log ||T_t - m||_{p\to p}.$$

In this talk, we concern the relation among $\{\gamma_{p\to q}\}$.

Contents:

1. Properties on $\gamma_{p\to q}$,
2. Relation between hypercontractivity and $\gamma_{p\to q}$,
3. Sufficient conditions for L^p-spectra to be p-independent,
4. Properties on spectra on L^p-spaces
   of operators symmetric on the L^2-space,
5. Example that $\gamma_{p\to p}$ depends on p.

Define for a linear operator A_p on $L^p(m)$,

$$\sigma_{\mathrm{p}}(A_p) := \{\lambda \in \mathbb{C};\ \lambda - A_p \text{ is not injective on } L^p(m)\}$$

$$\sigma_{\mathrm{c}}(A_p)$$

$$:= \{\lambda \in \mathbb{C};\ \lambda - A_p \text{ is injective, but is not onto map, and } \mathrm{Ran}(\lambda - A_p) \text{ is dense in } L^p(m)\}$$

$$\sigma_{\mathrm{r}}(A_p)$$

$$:= \{\lambda \in \mathbb{C};\ \lambda - A_p \text{ is injective, but is not onto map, and } \mathrm{Ran}(\lambda - A_p) \text{ is not dense in } L^p(m)\}$$

$$\rho(A_p) := \{\lambda \in \mathbb{C};\ \lambda - A_p \text{ is bijective on } L^p(m)\}$$

$\sigma_{\mathrm{p}}(A_p)$, $\sigma_{\mathrm{c}}(A_p)$, $\sigma_{\mathrm{r}}(A_p)$ and $\rho(A_p)$ are disjoint and their union is equal to $\mathbb{C}$.

1. Properties on $\gamma_{p\to q}$

Proposition

Let $p_1, p_2, q_1, q_2 \in [1, \infty]$.

Let $r_1, r_2 \in [1, \infty]$ such that $\exists\theta \in [0, 1]$ satisfying

$$\frac{1}{r_1} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1} \quad \text{and} \quad \frac{1}{r_2} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}.$$

Then,

$$\gamma_{r_1\to r_2} \geq (1-\theta)\gamma_{p_1\to p_2} + \theta\gamma_{q_1\to q_2}.$$

In particular, $s \mapsto \gamma_{1/s\to 1/s}$ on $[0, 1]$ is concave.

Theorem

The function $p \mapsto \gamma_{p\to p}$ on $[1,\infty]$ is continuous on $(1,\infty)$. If $\gamma_{p\to p} > 0$ for some $p \in [1,\infty]$, then $\gamma_{p\to p} > 0$ for all $p \in (1,\infty)$.

Remark

The function $\gamma_{p\to p}$ may not be continuous at $p = 1, \infty$.

Indeed, if m has the standard normal distribution and $\{T_t\}$ is the Ornstein-Uhlembeck semigroup, then $\gamma_{p\to p} = 1$ for $p \in (1,\infty)$, $\gamma_{p\to p} = 0$ for $p = 1, \infty$.

Let p^* be the conjugate exponent of p, i.e. $\dfrac{1}{p}+\dfrac{1}{p^*}=1$.

Theorem

Assume that $\{T_t\}$ is self-adjoint on $L^2(m)$.

Then, $\gamma_{p\to p}=\gamma_{p^*\to p^*}$ for $p\in[1,\infty]$

and $p\mapsto\gamma_{p\to p}$ is non-decreasing on $[1,2]$

and non-increasing on $[2,\infty]$.

In particular, the maximum is attained at $p=2$.

2. Relation between hypercontractivity and $\gamma_{p\to q}$

If there exist $p, q \in (1, \infty)$, $K \geq 0$ and $C > 0$ such that $p < q$ and

$$||T_K f||_q \leq C||f||_p, \quad f \in L^p(m),$$

then for any $p', q' \in (1, \infty)$ such that $p' < q'$, there exist $K' \geq 0$ and $C' > 0$ and

$$||T_{K'} f||_{q'} \leq C'||f||_{p'}, \quad f \in L^{p'}(m).$$

(If $C = 1$, we can choose $C' = 1$.)

In this talk, we call $\{T_t\}$ hyperbounded,

if there exist $p, q \in (1, \infty)$, $K \geq 0$ and $C > 0$ such that $p < q$ and

$$||T_K f||_q \leq C||f||_p, \qquad f \in L^p(m). \tag{1}$$

If (1) holds with $C = 1$ and some p, q, K,

then we call $\{T_t\}$ hypercontractive.

Theorem

The following conditions are equivalent:

1. $\{T_t\}$ is hyperbounded.
2. $\gamma_{p\to q} \geq 0$ for some $1 < p < q < \infty$.
3. $\gamma_{p\to q} = \gamma_{2\to 2}$ for all $p, q \in (1, \infty)$.

Proposition

$$||T_K f||_r \leq ||f||_2, \quad f \in L^2(m)$$

for some $K > 0$ and $r > 2$. Then, we have

$$||T_K f - \langle f \rangle||_2 \leq (r-1)^{-1/2} ||f||_2, \quad f \in L^2(m),$$

$$||T_t f - \langle f \rangle||_2 \leq \sqrt{r-1} \exp\left\{-\frac{t}{K} \log \sqrt{r-1}\right\} ||f||_2,$$

$$f \in L^2(m),\ t \in [0, \infty).$$

Theorem

The following conditions are equivalent:

1. $\{T_t\}$ is hypercontractive.
2. $\gamma_{p \to q} > 0$ for some $1 < p < q < \infty$.
3. $\gamma_{p \to q} = \gamma_{2 \to 2}$ for all $p, q \in (1, \infty)$ and $\gamma_{2 \to 2} > 0$.
4. There exist $K > 0$ and $r > 0$ such that

$$||T_K||_{2 \to r} < \infty \quad \text{and} \quad ||T_K - m||_{2 \to 2} < 1.$$

3. Sufficient conditions for L^p-spectra to be p-independent

Assume that $\{T_t\}$ is hyperbounded.

Let $\mathfrak{A}_p$ be the generator of $\{T_t\}$ on $L^p(m)$

for $p \in [1, \infty)$.

Assume that $\mathfrak{A}_2$ is a *normal* operator,

i.e. $(\mathfrak{A}_2)^* \mathfrak{A}_2 = \mathfrak{A}_2 (\mathfrak{A}_2)^*$.

In this section, we see that

the spectra of $\mathfrak{A}_p$ are independent of p.

Under the assumption, we can consider the spectral decomposition of $-\mathfrak{A}_2$ as follows:

$$-\mathfrak{A}_2 = \int_{\mathbb{C}} \lambda dE_\lambda.$$

For a bounded $\mathbb{C}$-valued measurable function ϕ on $\mathbb{C}$, define an operator $\phi(-\mathfrak{A}_2)$ on $L^2(m)$ by

$$\phi(-\mathfrak{A}_2) = \int_{\mathbb{C}} \phi(\lambda) dE_\lambda.$$

We can regard $\phi(-\mathfrak{A}_2)$ as a linear operator on $L^p(m)$.

Proposition

Let h be a $\mathbb{C}$-valued bounded measurable function on $\mathbb{C}$ which is analytic on the neighborhood around 0 and define $\phi(\lambda) := h(1/\lambda)$.

Then, $\phi(-\mathfrak{A})$ is a bounded operator on $L^p(m)$.

Theorem

Assume that $\{T_t\}$ is hyperbounded and $\mathfrak{A}_2$ is normal.

Then, $\sigma(-\mathfrak{A}_q) = \sigma(-\mathfrak{A}_2)$ for $q \in (1, \infty)$.

By a little more calculation, we have the following theorem.

Theorem

Assume that $\{T_t\}$ is hyperbounded and $\mathfrak{A}_2$ is normal. Then, $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \emptyset$ for $p \in (1, \infty)$.

If there exists positive constants K and C such that

$$||T_K f||_\infty \leq C||f||_1, \quad f \in L^1(m),$$

then $\{T_t\}$ is called ultracontractive.

Theorem

Assume that $\{T_t\}$ is ultracontractive and that $\mathfrak{A}_2$ is a normal operator. Then, $\sigma(-\mathfrak{A}_p) = \sigma(-\mathfrak{A}_2)$ for $p \in [1, \infty)$. Moreover, $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \emptyset$ for $p \in [1, \infty)$.

4. Properties on spectra on L^p-spaces of operators symmetric on the L^2-space

Let A_p be a densely defined, closed, and real operator on $L^p(m)$ for $p \in [1, \infty)$.
Assume that $\{A_p; p \in [1, \infty)\}$ are consistent,
i.e. if $p > q$, then $\mathrm{Dom}(A_p) \subset \mathrm{Dom}(A_q)$
and $A_p f = A_q f$ for $f \in \mathrm{Dom}(A_p)$.
A Markovian semigroup $\{T_t\}$ and its generators $\{\mathfrak{A}_p; p \in [1, \infty)\}$ satisfy the assumption on $\{A_p; p \in [1, \infty)\}$.

Additionally assume that A_2 is self-adjoint on $L^2(m)$, i.e. $A_2 = A_2^*$.

Lemma

$\sigma_{\mathrm{r}}(A_p) = \emptyset$ for $p \leq 2$.

Theorem

We have the following.

1. $\sigma_{\mathrm{p}}(A_p) \subset \sigma_{\mathrm{p}}(A_q)$ for $q \leq p$.
2. $\sigma_{\mathrm{r}}(A_q) \subset \sigma_{\mathrm{r}}(A_p)$ for $q \leq p$.
3. $\sigma_{\mathrm{c}}(A_p) \subset \sigma_{\mathrm{c}}(A_q) \cup \sigma_{\mathrm{p}}(A_q)$ for $q \leq p \leq 2$.
4. $\rho(A_q) \subset \rho(A_p)$ for $q \leq p \leq 2$.

$\sigma(A_p)$ is decreasing for $p \in [1, 2]$

and increasing for $p \in [2, \infty)$.

Corollary

Let $p \in [2, \infty)$. Then the followings hold.

1. $\sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{r}}(A_p) = \sigma_{\mathrm{p}}(A_{p^*})$.
2. $\sigma_{\mathrm{c}}(A_p) = \sigma_{\mathrm{c}}(A_{p^*})$.

Corollary

$\sigma_{\mathrm{p}}(A_p) \subset \mathbb{R}$ for $p \in [2, \infty)$.

Since A_2 is a self-adjoint operator, by using the general theory of self-adjoint operators on Hilbert spaces it is obtained that $\sigma(A_2) \subset \mathbb{R}$.
However, when $p \neq 2$, it does not always hold.

5. Example that $\gamma_{p\to p}$ depends on p

Let $p \in [1, \infty)$.

Define a measure ν on $[0, \infty)$ by $\nu(dx) := e^{-x}dx$

and a differential operator $\mathfrak{A}_p$ on $L^p(\nu)$ by

$$\mathrm{Dom}(\mathfrak{A}_p) := \left\{ f \in W^{2,p}(\nu; \mathbb{C}); f'(0) = 0 \right\},$$
$$\mathfrak{A}_p := \frac{d^2}{dx^2} - \frac{d}{dx}.$$

Note that $\mathfrak{A}_2$ is a self-adjoint operator on $L^2(\nu)$.

The self-adjointness on $L^2(\nu)$ implies

that $\{T_t\}$ is analytic semigroup on $L^p(m)$ for $p \in (1, \infty)$.

Let $p \in [1,2]$.

Consider the linear transformation I defined by

$$(If)(x) := e^{-x/2} f(x).$$

Then, we have

$$\int_0^\infty |If(x)|^p e^{(\frac{p}{2}-1)x} dx = \int_0^\infty |f(x)|^p \nu(dx),$$

and $f'(0) = 0$ if and only if $\frac{1}{2}(If)(0) + (If)'(0) = 0$.

Hence, I is an isometric transformation

from $L^p(\nu)$ to $L^p(\tilde{\nu}_p)$, where $\tilde{\nu}_p := e^{(\frac{p}{2}-1)x} dx$.

Define a linear operator $\tilde{\mathfrak{A}}_p$ on $L^p(\tilde{\nu}_p)$ by

$$\mathrm{Dom}(\tilde{\mathfrak{A}}_p) := \left\{ f \in W^{2,p}(\nu; \mathbb{C}); \frac{1}{2} f(0) + f'(0) = 0 \right\},$$

$$\tilde{\mathfrak{A}}_p := \frac{d^2}{dx^2} - \frac{1}{4}.$$

Then, we have the following commutative diagram.

$$\begin{array}{ccc} L^p(\nu) & \xrightarrow{\mathfrak{A}_p} & L^p(\nu) \\ I \downarrow & & \downarrow I \\ L^p(\tilde{\nu}_p) & \xrightarrow{\tilde{\mathfrak{A}}_p} & L^p(\tilde{\nu}_p) \end{array}$$

By this diagram we have

$$\sigma_{\mathrm{p}}(\mathfrak{A}_p) = \sigma_{\mathrm{p}}(\tilde{\mathfrak{A}}_p), \ \sigma_{\mathrm{c}}(\mathfrak{A}_p) = \sigma_{\mathrm{c}}(\tilde{\mathfrak{A}}_p), \ \sigma_{\mathrm{r}}(\mathfrak{A}_p) = \sigma_{\mathrm{r}}(\tilde{\mathfrak{A}}_p).$$

Hence, to see the spectra of $\mathfrak{A}_p$, it is sufficient to see the spectra of $\tilde{\mathfrak{A}}_p$.

From now we cannot discuss the cases that $1 \le p < 2$ and that $p = 2$ in the same way.

First we consider the case that $1 \le p < 2$.

Lemma

If $1 \le p < 2$, then

$$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p) = \{0\} \cup \left\{ x + iy;\ \ x, y \in \mathbb{R},\ \ x > \frac{p-1}{p^2}, \right.$$
$$\left. |y| < \left(\frac{2}{p} - 1\right) \sqrt{x - \frac{p-1}{p^2}} \right\}.$$

Proof. Let $\lambda \in \mathbb{C} \setminus \{\frac{1}{4}\}$. Then,

$$\begin{cases} -\dfrac{d^2}{dx^2}u + \dfrac{1}{4}u = \lambda u \\ \dfrac{1}{2}u(0) + u'(0) = 0, \end{cases}$$

if and only if

$$\begin{cases} u(x) = C_1 e^{x\sqrt{-\lambda+1/4}} + C_2 e^{-x\sqrt{-\lambda+1/4}} \\ C_1\left(1/2 + \sqrt{-\lambda + 1/4}\right) + C_2\left(1/2 - \sqrt{-\lambda + 1/4}\right) = 0. \end{cases}$$

For u satisfying above,

$$u \in L^p(\tilde{\nu}_p) \quad \text{if and only if}$$
$$\text{“}p\,\mathrm{Re}\sqrt{-\lambda + 1/4} + \frac{p}{2} - 1 < 0 \text{ or } C_1 = 0\text{”}.$$

($\sqrt{z} := \sqrt{r}e^{i\theta/2}$ for $z = re^{i\theta}$ where $r \geq 0,\ \theta \in (-\pi, \pi]$.)

Lemma

If $1 \le p < 2$, then

$$\rho(-\tilde{\mathfrak{A}}_p) \supset \left\{ x + iy;\ \ y^2 > \left(\frac{2}{p} - 1\right)^2 \left(x - \frac{p-1}{p^2}\right) \right\} \setminus \{0\}.$$

Proof. Let

$$\phi_\lambda(x) := \left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right) e^{x\sqrt{-\lambda+\frac{1}{4}}} - \left(\frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}}\right) e^{-x\sqrt{-\lambda+\frac{1}{4}}},$$

$$\psi_\lambda(x) := e^{-x\sqrt{-\lambda+\frac{1}{4}}},$$

$$W_\lambda := -2\sqrt{-\lambda + \frac{1}{4}} \left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right).$$

Define a $\mathbb{C}$-valued function g_λ on $[0,\infty) \times [0,\infty)$ by

$$g_\lambda(x,y) := \begin{cases} \dfrac{1}{W_\lambda}\phi_\lambda(x)\psi_\lambda(y), & x \le y \\ \dfrac{1}{W_\lambda}\phi_\lambda(y)\psi_\lambda(x), & y \le x \end{cases}$$

Let $G_\lambda f(x) := \int_0^\infty g_\lambda(x,y)f(y)dy$.

Then,

$$\left\{\lambda - (-\tilde{\mathfrak{A}}_p)\right\} G_\lambda f = f, \quad \text{and} \quad \frac{1}{2}G_\lambda f(0) + (G_\lambda f)'(0) = 0.$$

By checking the boundedness of G_λ on $L^p(\tilde{\nu}_p)$

we have the conclusion.

$$\tilde{\nu}_p = e^{(\frac{p}{2}-1)x}dx, \quad \tilde{\mathfrak{A}}_p = \frac{d^2}{dx^2} - \frac{1}{4},$$

$$\mathrm{Dom}(\tilde{\mathfrak{A}}_p) = \left\{ f \in W^{2,p}(\tilde{\nu}_p; \mathbb{C});\ \frac{1}{2} f(0) + f'(0) = 0 \right\}.$$

Theorem

Followings hold for $1 \le p < 2$.

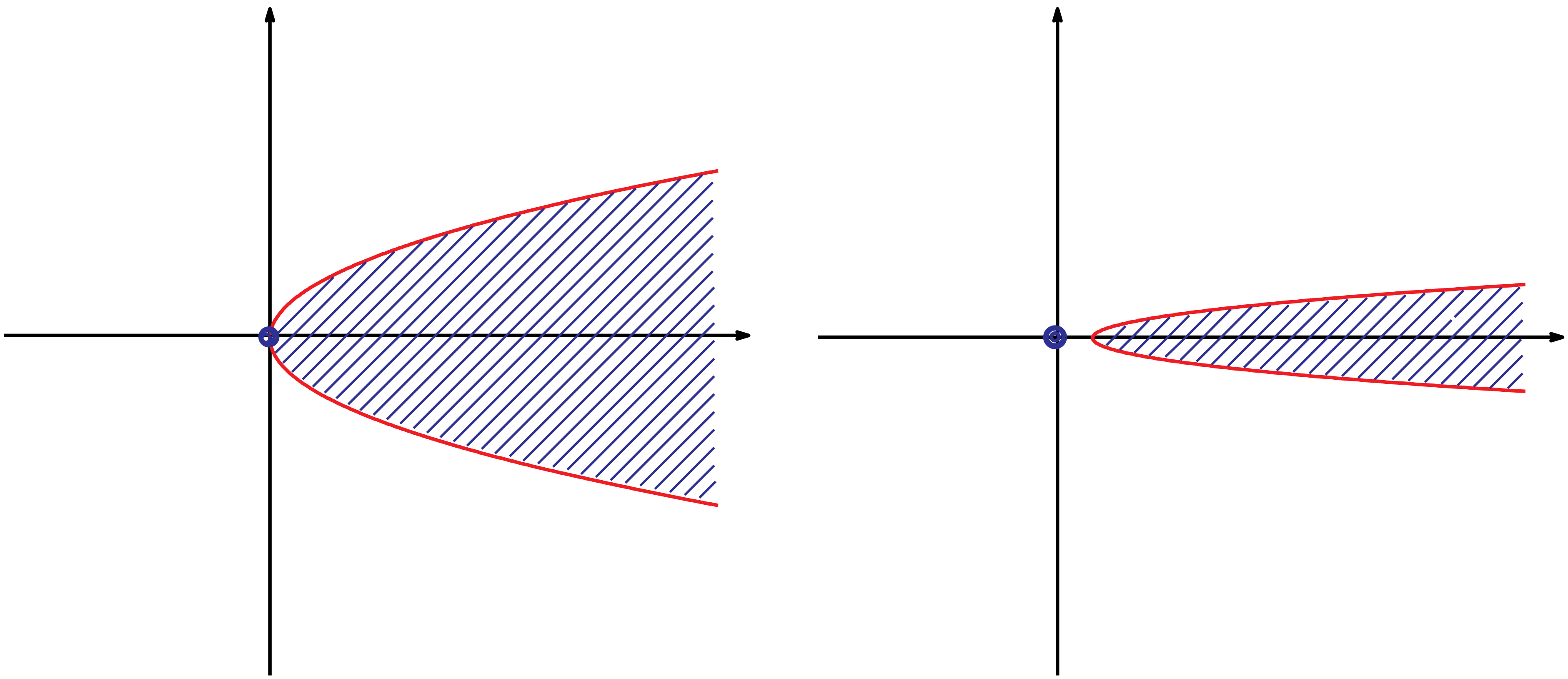
1. $$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p) = \{0\} \cup \left\{ x + iy;\ x, y \in \mathbb{R},\ x > \tfrac{p-1}{p^2} \text{ and } |y| < \left(\tfrac{2}{p} - 1\right)\sqrt{x - \tfrac{p-1}{p^2}} \right\},$$
2. $$\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_p) = \left\{ x + iy;\ x, y \in \mathbb{R},\ x \ge \tfrac{p-1}{p^2}, \text{ and } |y| = \left(\tfrac{2}{p} - 1\right)\sqrt{x - \tfrac{p-1}{p^2}} \right\} \setminus \{0\},$$
3. $$\rho(-\tilde{\mathfrak{A}}_p) = \left\{ x + iy;\ x, y \in \mathbb{R},\ y^2 > \left(\tfrac{2}{p} - 1\right)^2 \left(x - \tfrac{p-1}{p^2}\right) \right\} \setminus \{0\}.$$

$$\nu(dx) = e^{-x}dx, \quad \mathfrak{A}_p = \frac{d^2}{dx^2} - \frac{d}{dx},$$
$$\mathrm{Dom}(\mathfrak{A}_p) = \left\{ f \in W^{2,p}(\nu;\mathbb{C}); f'(0) = 0 \right\}.$$

Theorem

Followings hold for $1 \le p < 2$.

1. $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) = \{0\} \cup \Big\{ x + iy;\ x, y \in \mathbb{R},\ x > \frac{p-1}{p^2}$ and $|y| < \left(\frac{2}{p} - 1\right)\sqrt{x - \frac{p-1}{p^2}} \Big\},$
2. $\sigma_{\mathrm{c}}(-\mathfrak{A}_p) = \Big\{ x + iy;\ x, y \in \mathbb{R},\ x \ge \frac{p-1}{p^2}$ and $|y| = \left(\frac{2}{p} - 1\right)\sqrt{x - \frac{p-1}{p^2}} \Big\} \setminus \{0\},$
3. $\rho(-\mathfrak{A}_p)$ $= \left\{ x + iy;\ x, y \in \mathbb{R},\ y^2 > \left(\frac{2}{p} - 1\right)^2 \left(x - \frac{p-1}{p^2}\right) \right\} \setminus \{0\}.$



$p = 1$ $\qquad$ $1 < p < 2$

$\sigma_{\mathsf{p}}(-\mathfrak{A}_p)$: blue, $\quad \sigma_{\mathsf{c}}(-\mathfrak{A}_p)$: red

Next we check $\sigma(-\tilde{\mathfrak{A}}_2)$. Recall that

$$\tilde{\nu}_2 = dx, \quad \tilde{\mathfrak{A}}_2 = \frac{d^2}{dx^2} - \frac{1}{4},$$

$$\mathrm{Dom}(\tilde{\mathfrak{A}}_2) = \left\{ f \in W^{2,2}(dx; \mathbb{C});\ \frac{1}{2} f(0) + f'(0) = 0 \right\}.$$

Lemma $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2) = \{0\}$.

Now we check $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2)$.

$$\sigma_{\mathrm{disc}}(-\tilde{\mathfrak{A}}_2) := \{\lambda \in \sigma(-\tilde{\mathfrak{A}}_2);\ \lambda \text{ is isolated point of } \sigma(-\tilde{\mathfrak{A}}_2),$$
$$\lambda \text{ is an eigenvalue of finite multiplicity}\}$$

$$\sigma_{\mathrm{ess}}(-\tilde{\mathfrak{A}}_2) := \sigma(-\tilde{\mathfrak{A}}_2) \setminus \sigma_{\mathrm{disc}}(-\tilde{\mathfrak{A}}_2).$$

By the lemma above,

$$\sigma_{\mathrm{disc}}(-\tilde{\mathfrak{A}}_2) = \{0\}, \quad \sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2) = \sigma_{\mathrm{ess}}(-\tilde{\mathfrak{A}}_2).$$

Let $\tilde{\mathscr{E}}$ be the bilinear form associated with $\tilde{\mathfrak{A}}_2$. Then,

$$\tilde{\mathscr{E}}(f,g) = \int_0^\infty f'(x)g'(x)dx + \frac{1}{4}\int_0^\infty f(x)g(x)dx - \frac{1}{2}f(0)g(0).$$

Let

$$\tilde{\mathscr{E}}^{(0)}(f,g) = \int_0^\infty f'(x)g'(x)dx + \frac{1}{4}\int_0^\infty f(x)g(x)dx.$$

Then, $\tilde{\mathscr{E}}$ is a compact perturbation of $\tilde{\mathscr{E}}^{(0)}$.

Hence, by Weyl's theorem we have the following lemma.

Lemma

$$\sigma_{\text{ess}}(-\tilde{\mathfrak{A}}_2) = \sigma_{\text{ess}}(-\tilde{\mathfrak{A}}_2^{(0)}) = \left[\frac{1}{4}, \infty\right).$$

$$\tilde{\nu}_2 = dx, \quad \tilde{\mathfrak{A}}_2 = \frac{d^2}{dx^2} - \frac{1}{4},$$

$$\mathrm{Dom}(\tilde{\mathfrak{A}}_2) = \left\{ f \in W^{2,2}(dx;\mathbb{C});\ \frac{1}{2}f(0) + f'(0) = 0 \right\}.$$

Theorem

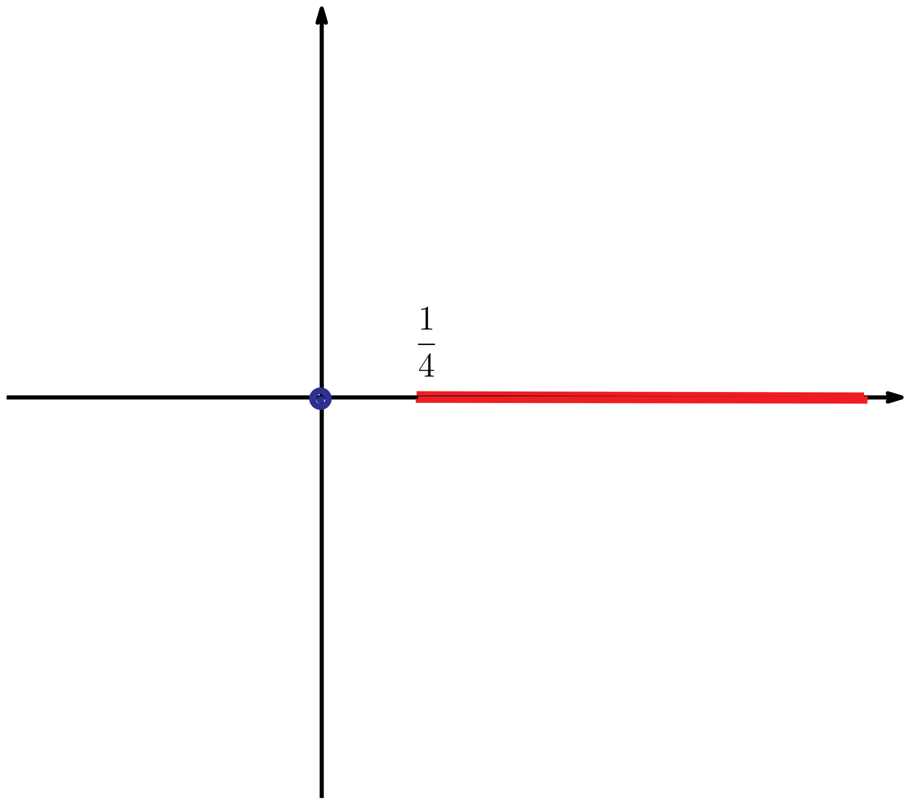
$$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2) = \{0\}, \quad \sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2) = \left[\frac{1}{4}, \infty\right).$$

$$\nu(dx) = e^{-x}dx, \quad \mathfrak{A}_p = \frac{d^2}{dx^2} - \frac{d}{dx},$$

$$\mathrm{Dom}(\mathfrak{A}_p) = \left\{ f \in W^{2,p}(\nu;\mathbb{C}); f'(0) = 0 \right\}.$$

Theorem

$$\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \{0\}, \quad \sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \left[\frac{1}{4}, \infty\right).$$



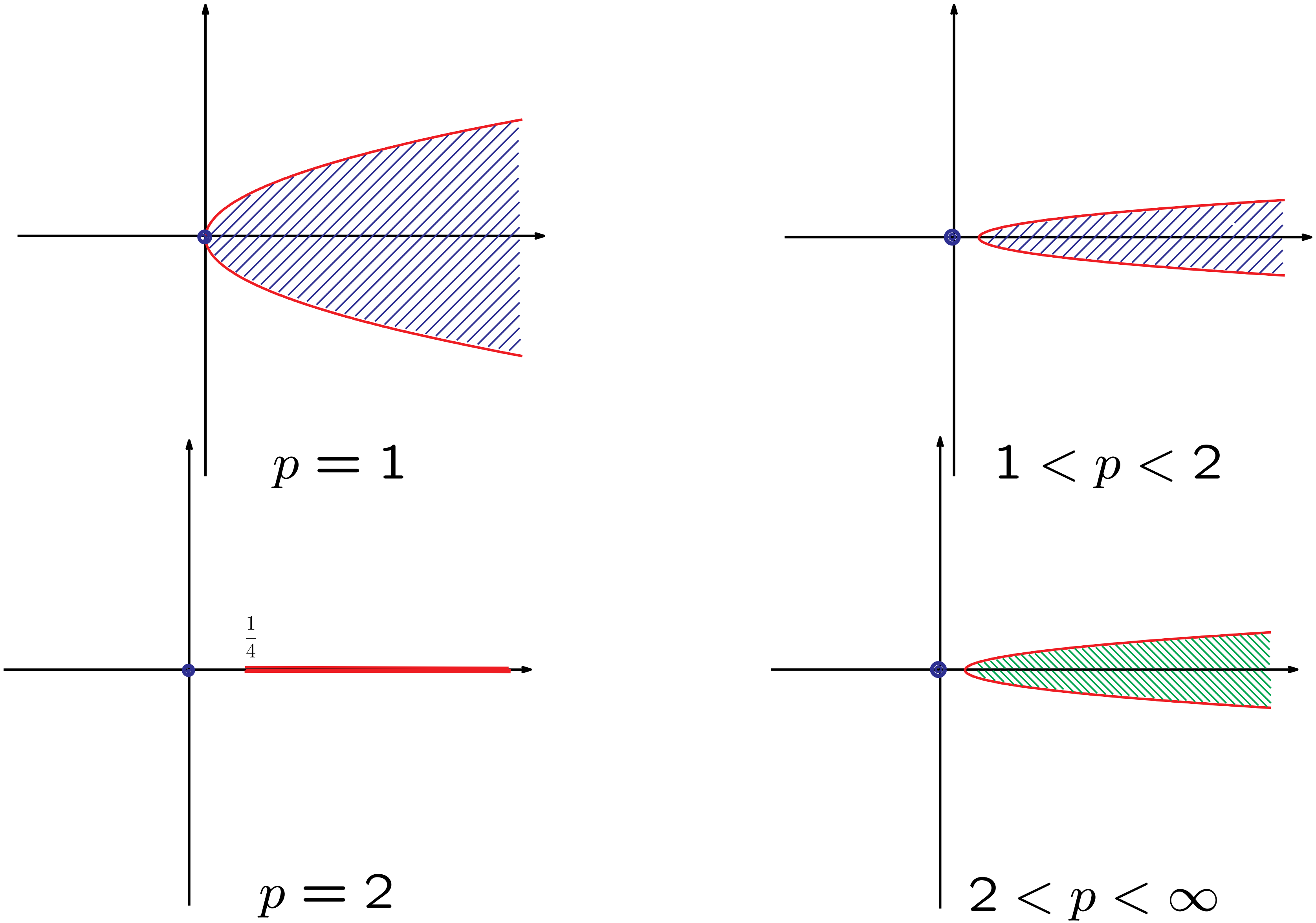
$$p = 2$$

$\sigma_{\mathsf{p}}(-\mathfrak{A}_2)$: blue, $\sigma_{\mathsf{c}}(-\mathfrak{A}_2)$: red

Theorem

For $p \in (2, \infty)$, we have the following.

1. $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) = \{0\}$,
2. $\sigma_{\mathrm{c}}(-\mathfrak{A}_p) = \left\{ x + iy;\ x, y \in \mathbb{R},\ x \geq \frac{p^* - 1}{p^{*2}} \right.$ $\left. \text{and } |y| = \left(\frac{2}{p^*} - 1 \right) \sqrt{x - \frac{p^* - 1}{p^{*2}}} \right\} \setminus \{0\}$,
3. $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \left\{ x + iy;\ x, y \in \mathbb{R},\ x > \frac{p^* - 1}{p^{*2}} \right.$ $\left. \text{and } |y| < \left(\frac{2}{p^*} - 1 \right) \sqrt{x - \frac{p^* - 1}{p^{*2}}} \right\}$,
4. $\rho(-\mathfrak{A}_p)$
$$= \left\{ x + iy;\ x, y \in \mathbb{R},\ y^2 > \left(\frac{2}{p^*} - 1 \right)^2 \left(x - \frac{p^* - 1}{p^{*2}} \right) \right\} \setminus \{0\}.$$



$\sigma_{\mathrm{p}}(-\mathfrak{A}_p)$: blue, $\sigma_{\mathrm{c}}(-\mathfrak{A}_p)$: red, $\sigma_{\mathrm{r}}(-\mathfrak{A}_p)$: green

Since $\{T_t\}$ is analytic on $L^p(m)$ for $p \in (1, \infty)$,

$$\sup\{\mathrm{Re}\lambda; \lambda \in \sigma(-\mathfrak{A}_p) \setminus \{0\}\} = -\lim_{t \to \infty} \frac{1}{t} \log ||T_t - m||_{p \to p}.$$

Hence, we obtain the following corollary.

Corollary

$$\gamma_{p \to p} = \frac{p-1}{p^2}, \ \ p \in [1, 2],$$
$$\gamma_{p \to p} = \frac{p^*-1}{(p^*)^2}, \ \ p \in [2, \infty].$$

Thank you for your attention!