Quenched invariance principle for random walks and random divergence forms in random media on cones

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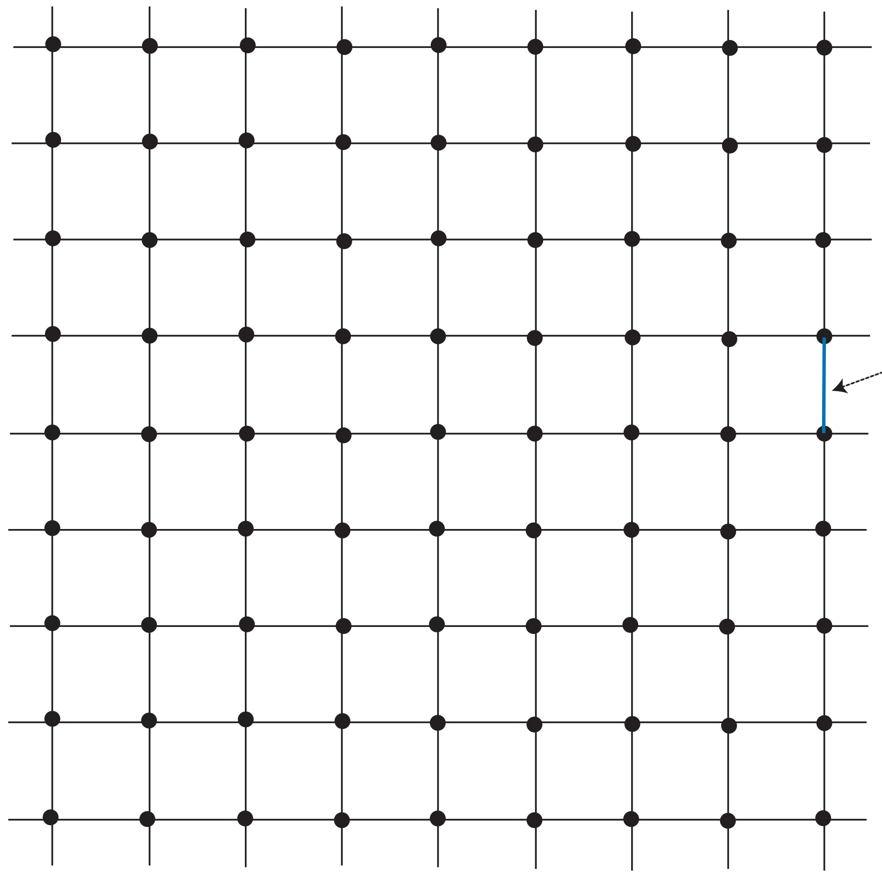
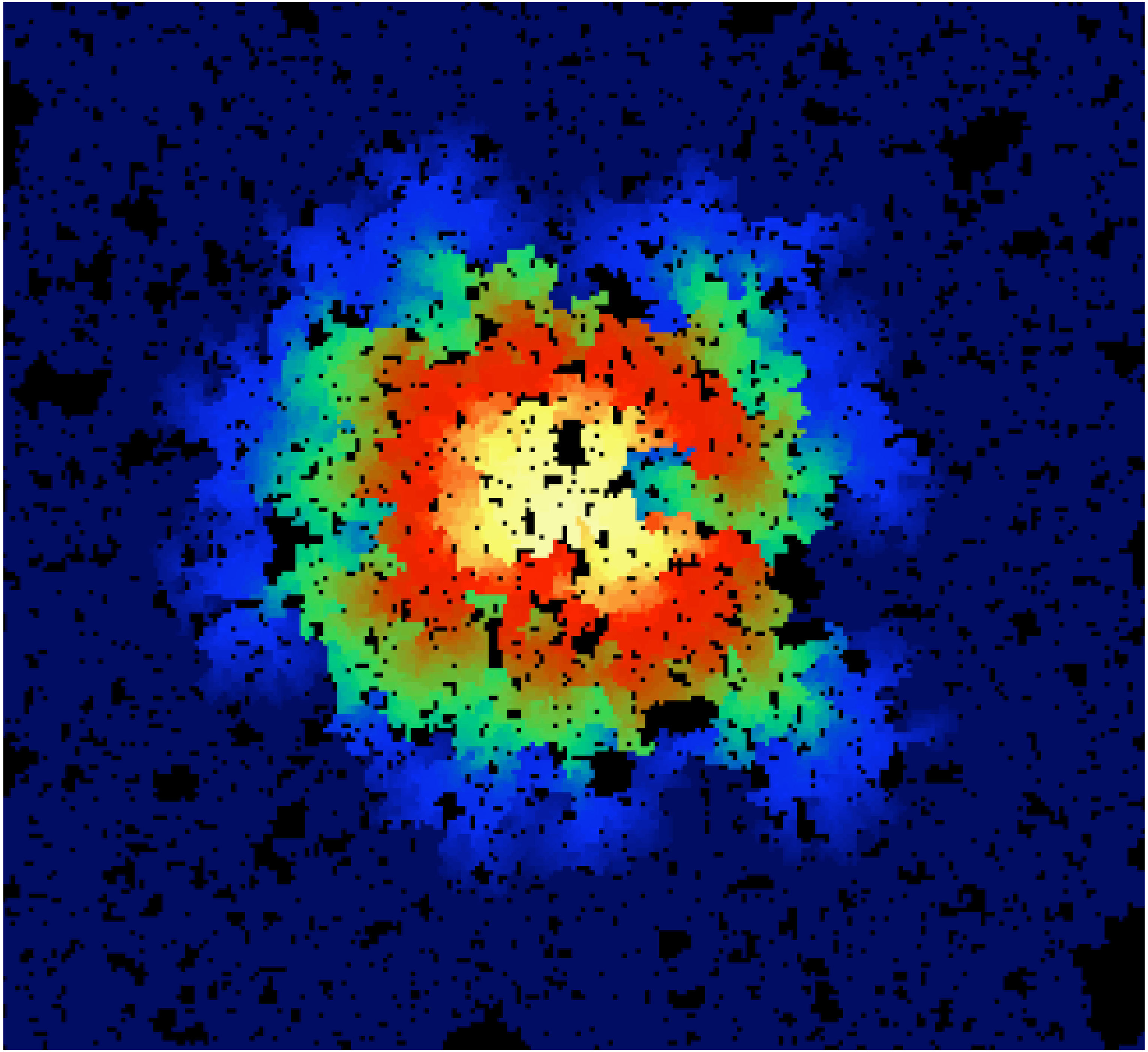
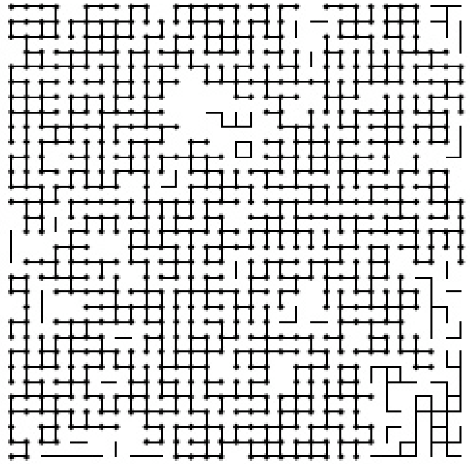
1 Introduction

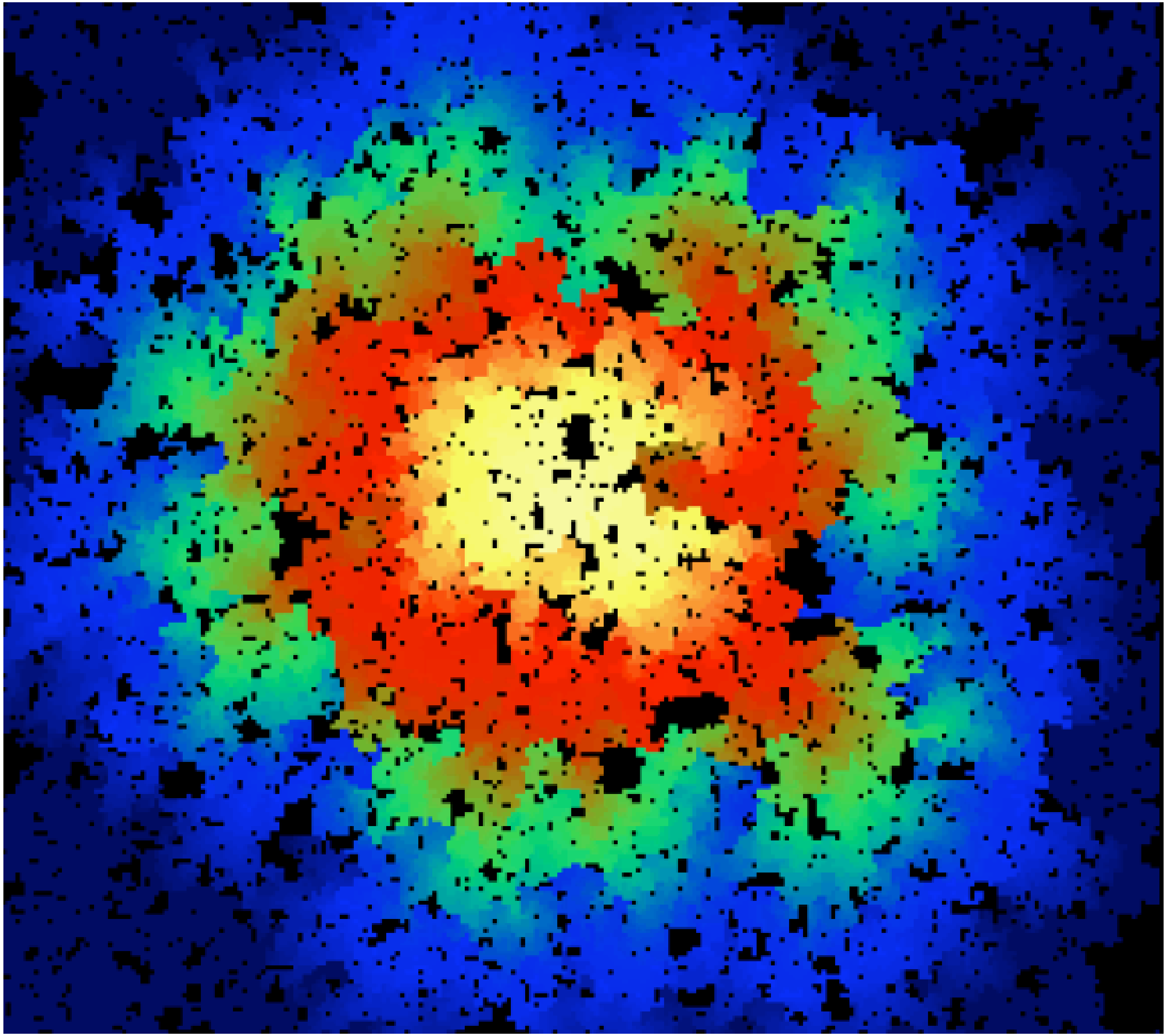
Bond percolation on $\mathbb{Z}^d\,(d \geq 2)$



$\exists p_c \in (0,1)$ s.t. $p > p_c \Rightarrow \exists 1\ \infty$-cluster $\mathcal{G}(\omega)$ (random media!), $p < p_c \Rightarrow$ no ∞-cluster







Known results SRW on supercritical percolation cluster on $\mathbb{Z}^d$

- [Quenched invariance principle (QIP)]

(Sidoravicius-Sznitman '04, Berger-Biskup '07, Mathieu-Piatnitski. '07)

$$n^{-1}Y^{\omega}_{n^2t} \to B_{\sigma t} \quad \mathbb{P}^*\text{-a.s. } \omega \text{ for some } \ \sigma > 0$$

- [Gaussian heat kernel bounds] (Barlow '04) $p^{\omega}_t(x,y) := \mathbb{P}^x(Y_t = y)/\mu_y$.

$$\frac{c_1}{t^{d/2}}\exp(-c_2\frac{d(x,y)^2}{t}) \le p^{\omega}_t(x,y) \le \frac{c_3}{t^{d/2}}\exp(-c_4\frac{d(x,y)^2}{t}),$$

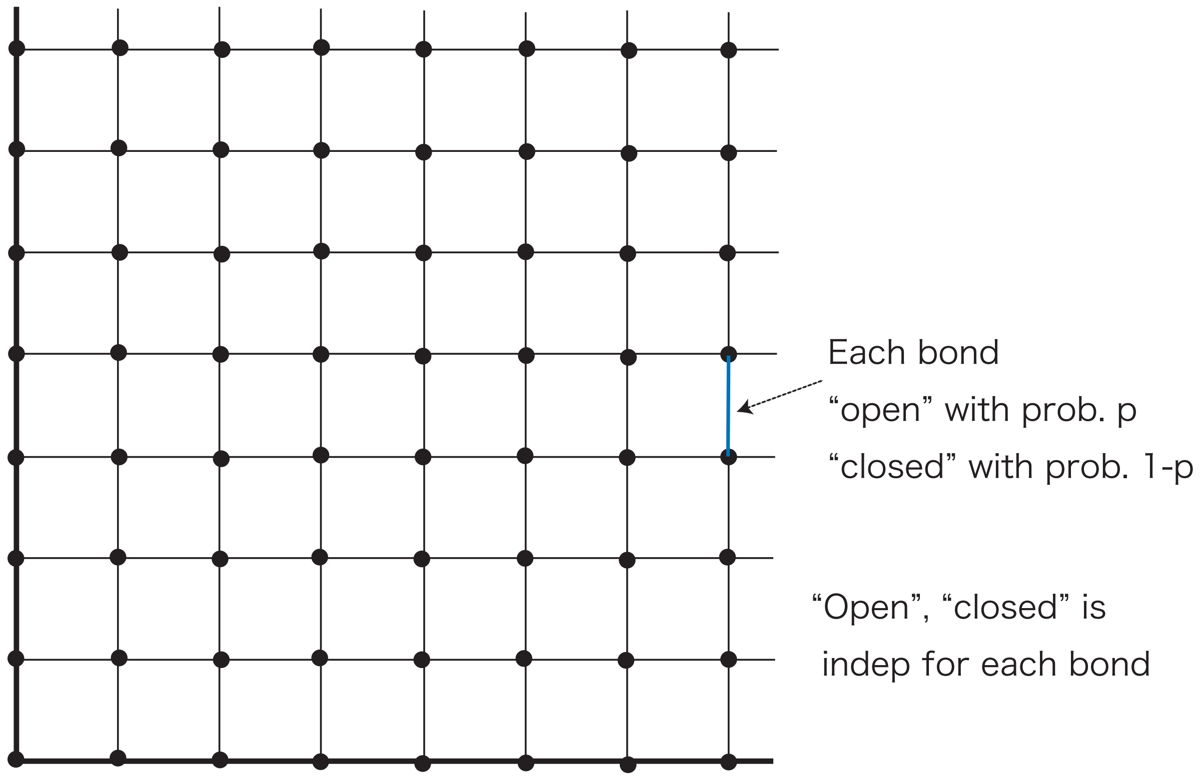
$\mathbb{P}^*$-a.s. ω for $t \ge d(x,y) \vee \exists U_x$, $x,y \in \mathcal{C}$.

Rem 1. "Annealed" invariance principle: known since 80's

Rem 2. Generalization of the QIP to random conductance model is known.

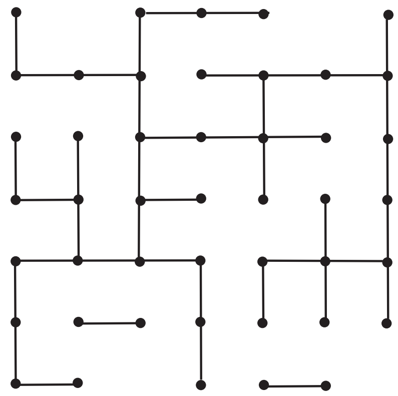
(Our problem) **Ex 1** RW on supercritical percolation cluster for $\mathbb{L} \subset \mathbb{Z}^d$ $(d \geq 2)$

$\mathbb{L} := \{(x_1, \cdots, x_d) \in \mathbb{Z}^d : x_{j_1}, \cdots, x_{j_l} \geq 0\}$ for some $1 \leq j_1 < \cdots < j_l \leq d$, $l \leq d$.



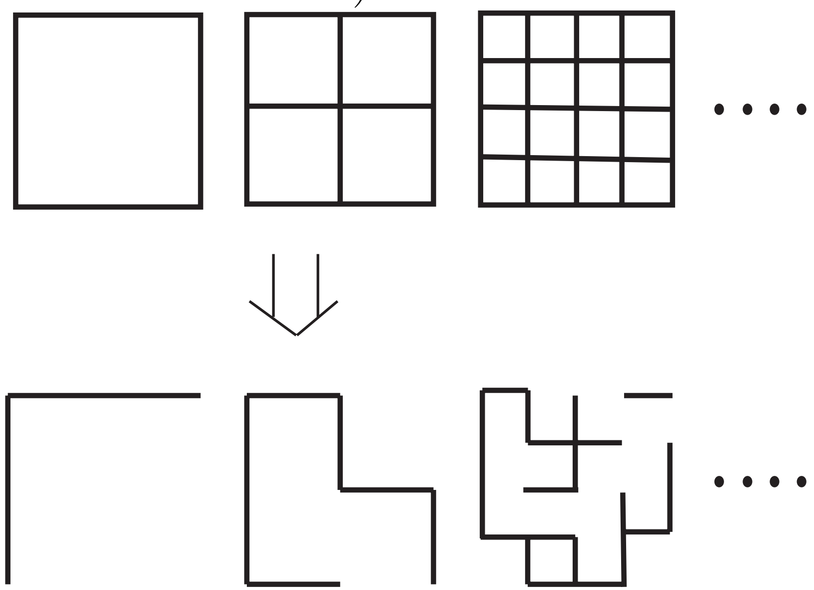
$\exists p_c \in (0,1)$ s.t. $\exists 1 \infty$-cluster $\mathcal{C}$ for $p > p_c$, no ∞-cluster for $p < p_c$.

$\mathcal{C}(\omega)$: ∞-cluster, $\mathbb{P}^*(\cdot) := \mathbb{P}_p(\cdot|0 \in \mathcal{C})$, Y^ω: SRW on $\mathcal{C}(\omega)$.



(Q1) $n^{-1}Y^\omega_{[n^2t]} \to B_{\sigma t}$, $\mathbb{P}^* -$ a.e. ω (for some $\sigma > 0$)?

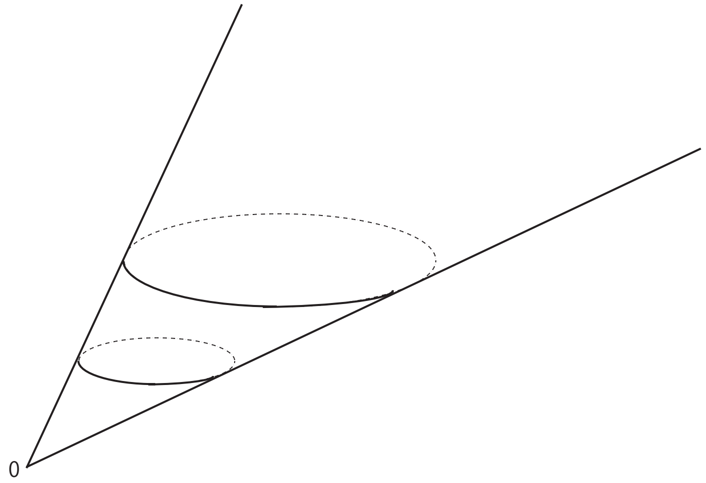
(How about RW on percolation on boxes?)



Ex 2 Random divergence form on a cone

C: Lipschitz domain in $\mathbb{R}^{d-1}$

$D := \{(t, tx_1, \cdots, tx_{d-1}) \in \mathbb{R}^d : t > 0, (x_1, \cdots, x_{d-1}) \in C\}$: cone



$(\Omega, \mathbb{P})$: Prob. space, $\omega \in \Omega$, $c_1 I \le A^\omega(x) \le c_2 I$ for all $x \in \overline{D}$, $\mathbb{P}$-a.e. ω.

$\exists \tilde{A}^\omega(x), \in \mathbb{R}^d$ s.t. $\tilde{A}^\omega(x) = A^\omega(x), x \in \overline{D}$, $\tilde{A}^\omega(x) = \tilde{A}^{\tau_x \omega}(0)$, $\{\tau_x\}_{x \in \mathbb{R}^d}$: ergo. shift.

$\mathcal{E}(f,f) = \int_D \nabla f(x) A^\omega(x) \nabla f(x) dx$ $\Rightarrow$ Y^ω: corresponding diffusion.

(Q2) $\varepsilon Y^\omega_{\varepsilon^{-2}t} \to B_{\sigma t}$, $\mathbb{P} -$ a.e. ω (for some $\sigma > 0$)?

(Known results for the whole space) Random divergence form on $\mathbb{R}^d$

$(\Omega, \mathbb{P})$: Prob. space, $\omega \in \Omega$, $\quad c_1 I \le A^\omega(x) \le c_2 I$ for all $x \in \mathbb{R}^d$, $\mathbb{P}$-a.e. ω,

$A^\omega(x) = A^{\tau_x \omega}(0)$, $\{\tau_x : x \in \mathbb{R}^d\}$: ergo. shift.

$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \nabla f(x) A^\omega(x) \nabla f(x) dx \quad \Rightarrow \quad Y^\omega$: corresponding diffusion.

- [Quenched invariance principle] (...., Osada '83, Kozlov '85)

$$\varepsilon Y^\omega_{\varepsilon^{-2}t} \to B_{\sigma t}, \quad \mathbb{P}\text{-a.s. } \omega \text{ for some } \sigma > 0.$$

- [Gaussian heat kernel bounds]

$$\frac{c_1}{t^{d/2}} \exp\left(-c_2 \frac{d(x,y)^2}{t}\right) \le p_t^\omega(x,y) \le \frac{c_3}{t^{d/2}} \exp\left(-c_4 \frac{d(x,y)^2}{t}\right), \qquad (1)$$

$\mathbb{P}$-a.s. ω for $t > 0$, $x, y \in \mathbb{R}^d$.

Problem in extending the results to cones

All the results use corrector method, which requires translation invariance of the original space.

Main results: Yes! **(Q1)** (box case as well) and **(Q2)** hold.

Ideas

- Full use of heat kernel estimates.
- Use information of QIP on the whole space and Dirichlet form methods.

2 Framework and results

$D \subset \mathbb{R}^d$: Lipschitz domain

$$\begin{aligned} \mathcal{E}(f,f) &= \frac{C}{2}\int_D |\nabla f(x)|^2 dx, \quad \forall f \in W^{1,2}(D), \\ W^{1,2}(D) &= \{f \in L^2(D,m) : \nabla f \in L^2(D,m)\}, \quad m: \text{ Lebesgue meas.} \end{aligned}$$

X: reflected BM corresponding to $(\mathcal{E}, W^{1,2}(D))$

X^D: process killed on exiting D (i.e. X^D corresponding to $(\mathcal{E}, W_0^{1,2}(D))$).

$\{D_n\}_{n\geq 1} \subset \overline{D}$: D_n supports a meas. m_n s.t. $m_n \to m$ weakly in $\overline{D}$.

Theorem 2.1 $\{X^n_t\}_{t\ge 0}$: *sym. Hunt proc. on* $L^2(D_n;m_n)$, $m_n \overset{\text{weak}}{\longrightarrow} m$ *on* $\overline{D}$.

Assume that $\forall\{n_j\}$ *subseq.,* $\exists\{n_{j_k}\}$ *sub-subseq. and*

$\exists(\widetilde{X},\widetilde{\mathbb{P}}^x, x\in\overline{D})$: *m-sym. conserv. conti. Feller proc. on* $\overline{D}$ *starting at* x *s.t.*

(i) $\forall x_j \to x$, $\mathbb{P}^{n_{j_k}}_{x_j} \Rightarrow \widetilde{\mathbb{P}}_x$ *weakly in* $\mathbb{D}([0,\infty),\overline{D})$,

(ii) $\widetilde{X}^D \overset{d}{=} X^D$ *where* $\widetilde{X}^D$ *is subprocess of* $\widetilde{X}$ *killed upon leaving* D,

(iii) $(\tilde{\mathcal{E}},\tilde{\mathcal{F}})$: *D-form of* $\widetilde{X}$ *on* $L^2(D;m)$ *satisfies*

$$\mathcal{C}\subset\tilde{\mathcal{F}} \qquad \text{and} \qquad \tilde{\mathcal{E}}(f,f)\le K\mathcal{E}(f,f)\ \ \forall f\in\mathcal{C},$$

where $\mathcal{C}$: *core for* $(\mathcal{E},W^{1,2}(D))$ *and* $K\ge 1$.

$\Rightarrow (X^n,\mathbb{P}^n_{x_n}) \overset{\text{weak}}{\to} (X,\mathbb{P}_x)$ *in* $\mathbb{D}([0,\infty),\overline{D})$ *as* $n\to\infty$.

How to verify (i)-(iii)?

(i) Use heat kernel esitmates etc.

(ii) From QIP of the whole space (iii) By LLN-type arguments

2.1 About (i)

Assume $0 \in D_n$, $\forall n \geq 1$, $\exists \delta_n \in [0,1]$ with $\lim_{n\to\infty} \delta_n = 0$ s.t. $|x-y| \geq \delta_n$ $\forall x \neq y \in D_n$.

Assumption 2.2 *(I)* $\exists c_1, c_2, c_3, \alpha, \beta, \gamma > 0$, $N_0 \in \mathbb{N}$ *s.t. the following hold for all* $n \geq N_0$, $x_0 \in B(0, c_1 n^{1/2}) \cap D_n$, *and all* $\delta_n^{1/2} \leq r \leq 1$.

(a) $E^x[\tau_{B(x_0,r)\cap D_n}(X^n)] \leq c_2 r^\beta$, $\forall x \in B(x_0, r/2) \cap D_n$, *where* $\tau_A := \{t \geq 0 : X_t \notin A\}$.

(b) **Ellip. Harnack:** $\forall h_n$*: bdd. in* D_n *and harm. (w.r.t.* X^n*) in* $B(x_0, r)$*, then*

$$|h_n(x) - h_n(y)| \leq c_3 \left(\frac{|x-y|}{r}\right)^\gamma \|h_n\|_\infty \qquad \text{for all} \quad x, y \in B(x_0, r/2). \tag{2}$$

(II) $\forall\{x_n \in D_n : n \geq 1\}$ *and* $\forall x \in \overline{D}$ *s.t.* $x_j \longrightarrow x \in \overline{D}$, $\{\mathbb{P}_n^{x_n}\}_n$ *is tight in* $\mathbb{D}(\mathbb{R}_+, \overline{D})$.

(III) $J(X) := \int_0^\infty e^{-u}\{1 \wedge (\sup_{0\leq t\leq u}|X_t - X_{t-}|)\}du \stackrel{d}{\to} 0$.

Proposition 2.3 *Under Assumption 2.2, the following holds:*

$\forall\{n_j\}$ subseq., $\exists\{n_{j_k}\}$ sub-subseq. and m-sym. diffusion $(\widetilde{X}, \widetilde{\mathbb{P}}^x, x \in \overline{D})$ on $\overline{D}$ s.t. $\forall x_j \to x$, $\mathbb{P}_{n_{j_k}}^{x_j} \stackrel{\text{weak}}{\to} \widetilde{\mathbb{P}}^x$ in $\mathbb{D}([0,\infty), \overline{D})$.

Rem. Roughly, Gaussian-type heat kernel est. are enough to verify Assumption 2.2.

(Obtain equi-Hölder cont. for resolvents and use Ascoli-Arzela etc.:

$$|U_n^\lambda f(x) - U_n^\lambda f(y)| \leq C d(x,y)^{\gamma'}\|f\|_\infty, \text{where } U_n^\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t^n)\, dt.$$

For the case of random media, use Borel-Cantelli as well.)

3 Answer to (Q2) in Example 2

- Condition (ii): Whole space QIP by Osada, Kozlov $\Rightarrow$ (ii) holds.
- Condition (i): By uniform ellipticity,

$$\mathcal{E}(f,f) = \int_D \nabla f(x) A^\omega(x) \nabla f(x) dx \asymp \int_D |\nabla f(x)|^2 dx, \quad \forall f \in W^{1,2}(D). \tag{3}$$

$\Rightarrow \mathcal{E}^\varepsilon$: D-form corresp. to $\varepsilon Y^\omega_{\varepsilon^{-2}\cdot}$ also satisfies (3). (Note: $\varepsilon D = D$.)

$\Rightarrow$ Gaussian-type HK est. (1) still holds uniformly for $\mathcal{E}^\varepsilon$.

(Due to the stability: (1) $\Leftrightarrow$ (Vol. doubling)+ (Poincaré ineq.) i.e.)

$$\mu(B(x_0,2R)\cap D) \;\le\; C_1 \mu(B(x_0,R)\cap D),$$

$$\int_{B(x_0,R)\cap D} (f(x) - \overline{f}_B)^2 \mu(dx) \;\le\; C_2 R^2 \int_{B(x_0,2R)\cap D} |\nabla f(x)|^2 dx, \quad \forall f \in W^{1,2}(D),$$

$\forall x_0 \in D, R > 0$ where $\overline{f}_B = \int_B f(x)\mu(dx)$.

$\Rightarrow$ Assumption 2.2 holds.

- Condition (iii): Any subsequ. limit $\tilde{\mathcal{E}}$ still satisfies (3) $\Rightarrow$ (iii) holds.

4 Answer to (Q1) in Example 1

- Condition (ii): Whole space QIP by Berger-Biskup, Mathieu-Piatnitski $\Rightarrow$ (ii) holds.
- Condition (iii): LLN-type arguments as follows:

Lemma 4.1 *$\{\eta_i\}_i$: i.i.d. with $E|\eta_1| < \infty$.*

$\{a_k^n\}_{k=1}^n$: $a_k^n \in \mathbb{R}$, $|a_k^n| \le M$ $\forall k, n$, $a := \exists \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n a_k^n$, $\exists \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n |a_k^n|$.

$\Rightarrow$ $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n a_k^n \eta_k = aE[\eta_1]$ *almost surely.*

Let $\mu_x^\omega = (\sharp \text{ of bonds in } \mathcal{C} \text{ con. to } x)$, $D_n = n^{-1}\mathbb{L}$, $\overline{D} = \{(x_1, .., x_d) \in \mathbb{R}^d : x_{j_1}, .., x_{j_l} \ge 0\}$.

Proposition 4.2 *Let $\mathcal{E}^{(n)}$ be D-form corresp. to $n^{-1}Y^\omega_{[n^2\cdot]}$.*

$$\tilde{\mathcal{E}}(f,f) \le \lim_{n\to\infty} \mathcal{E}^{(n)}(f,f) = 2^{d-l}pd \int_{\overline{D}} |\nabla f(x)|^2 dx, \qquad \forall f \in C_c^2(\overline{D}), \tag{4}$$

$$\lim_{n\to\infty} \sum_{x\in D_n} f(x)\frac{\mu_{nx}^\omega}{n^d} = 2^{d-l}pd \int_{\overline{D}} f(x)dx, \qquad \forall f \in C_c(\overline{D}). \tag{5}$$

Proof of 1st ineq. of (4) :

$$\begin{aligned}\tilde{\mathcal{E}}(f,f) &= \sup_{t>0}\frac{1}{t}(f-P_t f,f) = \sup_{t>0}\liminf_{n_j\to\infty}\frac{1}{t}(f-P_t^{n_j}f,f)\\ &\le \liminf_{n_j\to\infty}\sup_{t>0}\frac{1}{t}(f-P_t^{n_j}f,f) = \liminf_{n_j\to\infty}\mathcal{E}^{(n_j)}(f,f).\end{aligned}$$

For the 2nd ineq., suppose Supp $f \subset B(0,M)\cap\overline{D}$. Then

$$\mathcal{E}^{(n)}(f,f) = \frac{n^{2-d}}{2}\sum_{x,y\in D_n, x\sim y}(f(x)-f(y))^2\mu_{nx,ny} = \frac{1}{n^d}\sum_{(x,y)\in H_{n,f}} n^2(f(x/n)-f(y/n))^2\mu_{x,y},$$

where $H_{n,f} := \{(x,y) : x,y\in\mathbb{L}\cap B(0,nM), x\sim y\}$. Note $\sharp M_{n,f}\sim 2^{d-l}d(nM)^d$.

Let $a^n_{(x,y)} := n^2(f(x/n)-f(y/n))^2 \in [0,\exists M']$ and $\eta_{(x,y)} := \mu_{x,y}$. By IP of SRW,

$$\lim_{n\to\infty}(2^{d-l}d(Mn)^d)^{-1}\sum_{(x,y)\in H_{n,f}} a^n_{(x,y)} = M^{-d}\int_{\overline{D}}|\nabla f(x)|^2dx.$$

So by Lemma 4.1,

$$\lim_{n\to\infty} n^{-d} \sum_{(x,y)\in H_{n,f}} a^n_{(x,y)}\eta_{(x,y)} = 2^{d-l}dp \int_{\overline{D}} |\nabla f(x)|^2 dx.$$

Proof of (5):

$$\sum_{x\in D_n} f(x)\frac{\mu_{nx}}{n^d} = n^{-d} \sum_{(x,y)\in H_{n,f}} f(x/n)\mu_{x,y},$$

$$\lim_{n\to\infty} n^{-d} \sum_{(x,y)\in H_{n,f}} f(x/n)_\pm = 2^{d-l}d \int_{\overline{D}} f(x)_\pm dx.$$

So by Lemma 4.1, we obtain (5). □

- Condition (i): Strategy (Percolation est.) $\Rightarrow$ (HK estimates) $\Rightarrow$ Assumption 2.2

Lemma 4.3 (Percolation est.) $\exists c_1, c_2, c_3 > 0$ *s.t.* $\forall x, y \in \mathbb{L}$,

$$\mathbb{P}\left(x, y \in \mathcal{C} \text{ and } d(x,y) \le c_1|x-y|\right) \le c_2 e^{-c_3|x-y|},$$

$$\mathbb{P}\left(x, y \in \mathcal{C} \text{ and } d(x,y) \ge c_1^{-1}|x-y|\right) \le c_2 e^{-c_3|x-y|},$$

where $|\cdot - \cdot|$ *is the Euclidean dist. and* $d(\cdot,\cdot)$ *is the graph dist.*

Rem. $\mathbb{Z}^d$ case by Antal-Pisztora and we exteded to the case of $\mathbb{L}$.

NB: This is **the only** place where we need the restriction to half/square spaces.

Theorem 4.4 (HK est.) *$\exists \delta, c_1, .., c_7 > 0$ and c_i s.t. the following holds.*

$\exists \Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ and $S_x, x \in \mathbb{L}$ s.t. $S_x(\omega) < \infty$, $\forall \omega \in \Omega_1$, $\forall x \in \mathcal{C}(\omega)$, and

$$\mathbb{P}(S_x \geq n, x \in \mathcal{C}) \leq c_1 e^{-c_2 n^\delta}.$$

(a) For $x, y \in \mathcal{C}(\omega)$ the transition density of Y satisfies

$$q_t^Y(x,y) \leq c_3 t^{-d/2} \exp(-c_4 |x-y|^2/t), \quad t \geq |x-y| \vee S_x, \tag{6}$$

$$q_t^Y(x,y) \geq c_5 t^{-d/2} \exp(-c_6 |x-y|^2/t), \quad t \geq |x-y|^{3/2} \vee S_x. \tag{7}$$

(b) Further, if $x \in \mathcal{C}(\omega)$, $t \geq S_x$ and $B = B(x, 2\sqrt{t})$ then

$$q_t^{Z,B}(x,y) \geq c_7 t^{-d/2}, \quad \text{for } y \in B(x, \sqrt{t}).$$

Rem. Given Lemma 4.3, we can prove similarly to Barlow ('04)

(also similarly to Andres-Barlow-Deuschel-Hambly ('11)).

5 Proof of Theorem 2.1

Since both are Feller, enough to show $(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}) = (\mathcal{E}, W^{1,2}(D))$.

$\widetilde{X}$: diffusion, no killings $\Rightarrow$ D-form is strong local $\Rightarrow$ $\widetilde{\mathcal{E}}(u,u) = \frac{1}{2}\widetilde{\mu}^c_{\langle u\rangle}(\overline{D})$, $\forall u \in \widetilde{\mathcal{F}}$.

Condition (iii) $\Rightarrow$ $W^{1,2}(D) \subset \widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{E}}(f,f) \le K\mathcal{E}(f,f)$, $\forall f \in W^{1,2}(D)$.

$\Rightarrow \widetilde{\mu}_{\langle u\rangle}(dx) \le \frac{CK}{2}|\nabla u(x)|^2 dx$, $\forall u \in W^{1,2}(D)$. So

$$\widetilde{\mu}_{\langle u\rangle}(\partial D) = 0, \quad \forall u \in W^{1,2}(D). \tag{8}$$

Condition (ii) $\oplus$ strong locality of $\widetilde{\mu}_{\langle u\rangle}$ $\Rightarrow$ functions in $\widetilde{\mathcal{F}}_b$ is locally in $W^{1,2}_0(D)$ and

$$1_D(x)\tilde{\mu}_{\langle u\rangle}(dx) = 1_D(x)\frac{C}{2}|\nabla u(x)|^2 dx, \quad \forall u \in \widetilde{\mathcal{F}}_b. \tag{9}$$

(8)+(9) $\Rightarrow$ $\widetilde{\mathcal{E}}(u,u) = \mathcal{E}(u,u)$, $\forall u \in W^{1,2}(D)$.

(9) $\Rightarrow$ $\forall u \in \widetilde{\mathcal{F}}_b$, $\int_D |\nabla u(x)|^2 dx < \infty$ so $u \in W^{1,2}(D) \Rightarrow \widetilde{\mathcal{F}} \subset W^{1,2}(D)$

So we obtain $(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}) = (\mathcal{E}, W^{1,2}(D))$. □

6 Remark and Generalization

Remark: Since we have heat kernel estimates and QIP, we have the followig LCLT:

$$\lim_{n\to\infty} \sup_{x\in\mathbb{L}} \sup_{t\geq T} |n^{d/2} q^{\omega}_{nt}(0,[n^{1/2}x]) - k_t(x)| = 0, \quad \mathbb{P}-\text{a.s.},$$

where $k_t(x) = (2\pi t\sigma^2)^{-d/2}\exp(-|x|^2/(2\sigma^2 t))$, $T > 0$, and $[x] := ([x_1], \cdots, [x_d])$.

Generalization to random conductance mocdel

We can prove QIP on $\mathbb{L}$ for the following RCM:

$$\mathbb{P}(\mu_e \in \{0\} \cup [c,\infty)) = 1 \ \text{ for } \exists c\ , \quad \mathbb{P}(\mu_e > 0) > p_c(\mathbb{Z}^d), \quad \mathbb{E}[\mu_e] < \infty.$$

For RCM bdd from above but NOT below, anomalous HK decay may occur

(Berger-Biskup-Hoffman-Kozma '08).