Stochastic completeness of jump processes and random walks

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Outline

Introduction

Volume growth criteria

Weighted graphs and metric graphs
A cluster of related objects

- \((X, d)\): a separable metric space such that all metric balls
  \[ B(x, r) = \{ y \in X : d(x, y) \leq r \} \]
  are compact;
- \(\mu\): a Radon measure with full support on \(X\);
- \((\mathcal{E}, \mathcal{F})\): a regular Dirichlet form (symmetric);
  e.g. \(\mathcal{F} = H^1(\mathbb{R}^n)\), \(\mathcal{E}(u, v) = \int_X (\nabla u \cdot \nabla v) \, dm\)
- \(\Delta\): nonnegative definite generator;
- \((P_t)_{t>0}\): Markovian semigroup;
- \((X_t)_{t \geq 0}\): Hunt process.
Typical examples

Beurling-Deny: for $u \in \mathcal{F} \cap C_c(X)$

$$\mathcal{E}(u, u) = \mathcal{E}^{(c)}(u, u) + \int_{X \times X - \text{diag}} (u(x) - u(y))^2 J(dx, dy)$$

$$+ \int_X u(x)v(x)k(dx),$$

- Brownian motion on a manifold, diffusions on metric graphs, $\rightarrow$ strongly local Dirichlet forms;
- $\alpha$-stable process on $\mathbb{R}^n$, random walks on weighted graphs, $\rightarrow$ jump type process on metric spaces;
- jump-diffusion processes.
Stochastic (in)completeness

Various points of view

1. process: infinite lifetime almost surely;
2. process: upper escape rate, “forefront”;
3. semigroup or heat kernel: \( P_t 1 = 1, \int_X p_t(x, y)\mu(dy) = 1; \)
4. heat equation: nonnegative bounded solutions to

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} u(x, t) + \Delta u(x, t) = 0, \\
u(\cdot, 0) \equiv 0;
\end{array} \right. \tag{1.1}
\]
Various points of view (continued)

5. “generator”: nonnegative bounded solutions to
\[ \Delta u + \lambda u = 0, \text{ or } \Delta u + \lambda u \leq 0; \]

6. “generator” (weak Omori-Yau): \( \Delta u \leq -\alpha \) on \( \Omega_\alpha = \{ x \in X : u(x) > \sup u - \alpha \} \);

7. Dirichlet form: \( \exists \{ u_n \} \subset \mathcal{F} \text{ with } 0 \leq u_n \leq 1, \lim_{n \to \infty} u_n = 1, \text{ s.t. } \lim_{n \to \infty} \mathcal{E}(u_n, v) = 0, \forall v \in L^1 \cap \mathcal{F}; \)

8. large scale geometry: “very negative” curvature \( \Rightarrow \) stochastic incompleteness;

9. large scale geometry: “not very large” volume growth \( \Rightarrow \) stochastic completeness.
Riemannian manifold case:

- Grigor’yan (through heat equation): geodesic metric \( d \), Riemannian volume \( \mu \),

\[
\int_{\infty}^{\infty} \frac{rdr}{\ln (\mu (B_d(x_0, r)))} = \infty, \tag{\diamond}
\]

implies stochastic completeness;

- special case: \( \mu (B_d(x_0, r)) \leq \exp(CR^2) \);
Riemannian manifold case (continued):

- Gaffney, Hsu, Karp-Li, Takeda, Davies, Pigola-Rigoli-Setti, Takegoshi through different approaches;

- Hsu and Qin: upper rate function,

$$t = \int \phi(t) \frac{rdr}{\ln (\mu (B_d(x_0, r)))) + \ln \ln r}; \quad (♣)$$

Volume growth for stochastic completeness (♦) and escape rate (♣): sharp for model manifolds.
Strongly local case (Sturm):

- “calculus” through energy measure: \( d\Gamma(u, u) \approx |\nabla u|^2 d\mu \);
- intrinsic metric \( \rho \) to replace \( d \):
  \[
  \rho(x, y) = \sup \{ u(x) - u(y) : u \in \mathcal{F}_{loc} \cap C(X), \quad d\Gamma(u, u) \leq d\mu \}.
  \]

- Assumption: \((X, \rho) \simeq (X, d)\);
- stochastically complete if
  \[
  \int_{\ln(\mu(B_{\rho}(x_0, r)))}^{\infty} rdr \ln(\mu(B_{\rho}(x_0, r))) = \infty, \quad (\$)
  \]

Key feature: \( d\Gamma(\rho(x, \cdot), \rho(x, \cdot)) \leq d\mu \).
Jump process case

Adapted metrics (Masamune-Uemura):

\[
\sup_{x \in X} \int_{X \setminus \{x\}} \left(1 \wedge d^2(x, y)\right) J(x, y) \mu(\text{d}y) = M < \infty. \quad (\heartsuit)
\]

Remark 2.1
related: Lévy process, Takeda, Frank-Lenz-Wingert

Example 2.2

\(\alpha\)-stable processes \((\alpha \in (0, 2))\) on \(\mathbb{R}^n\):

\[
J(x, y) = \frac{c_{n, \alpha}}{|x - y|^{n+\alpha}},
\]

Remark 2.3

(\heartsuit) is an analogue to \(|\nabla d(x, \cdot)| \leq 1\) in the manifold case.
Jump process case:

- **Masamune-Uemura**: for any $\varepsilon > 0$
  \[ e^{-\varepsilon d(x_0, x)} \in L^1(X, \mu), \]

- **Grigor’yan-H.-Masamune**:
  \[ \liminf_{r \to \infty} \frac{\log \mu(B_d(x_0, r))}{r \log r} < \frac{1}{2}; \]
Jump process case:

- Masamune-Uemura-Wang (jump-diffusion):

\[
\liminf_{r \to \infty} \frac{\log \mu (B_d(x_0, r))}{r \log r} < \infty;
\]

- Shiozawa-Uemura, Shiozawa: more general coefficients, \(d\) not necessarily a metric but a reasonable “length”.
Basic strategy for jump process:

- truncation and stability, \( c > 0 \) (jump size):
  \[ J'(x, y) = J(x, y)1_{d(x, y) \leq c}. \]

- Davies’ method: stochastic completeness \( \iff \)
  \[
  \lim_{n \to \infty} \langle f - P_t f, g_n \rangle = 0
  \]
  for any \( f \in Lip_c(X) \), where \( \{g_n\} \subset L^2 \cap L^\infty(X, \mu), \)
  \( 0 \leq g_n \uparrow 1. \)

- Davies’ method: estimate
  \[
  \langle u_t - f, g_n \rangle^2 = \left( \int_0^t \mathcal{E}(u_s, g_n)ds \right)^2
  \]
Basic difficulty: lack of a chain rule due to non-locality.

Example 2.4

Let $\psi(x) = \exp(\alpha d(x, x_0))$,

$$|\psi(x) - \psi(y)| \leq \alpha d(x, y) \exp(\alpha c) \psi(x).$$

The function $1/\psi$ is expected to compensate the volume growth.

Open problem: volume growth criterion (♦) and escape rate (♣)?
Weighted graphs:

- \((V, E)\): a simple graph;
- \(\omega : V \times V \to [0, \infty)\) as jump kernel \(J(dx, dy)\)
  1. \(\omega(x, y) = \omega(y, x)\) for all \(x, y \in V\);
  2. \((x, y) \in E \iff \omega(x, y) > 0\);
- \(\mu : V \to (0, \infty)\) as a Radon measure.

**Example 3.1**

1. “normalized”: \(\omega = 1_E, \mu = \text{deg}\);
2. “physical” (Weber, Wojciechowski): \(\omega = 1_E, \mu \equiv 1\).
The graph metric $d_0$:

$$d_0(x, y) = \inf\{n : \exists \text{ a path of length } n \text{ connecting } x, y\}. $$

The regular Dirichlet form:

$$\mathcal{E}(u, u) = \frac{1}{2} \sum_x \sum_y \omega(x, y) (u(x) - u(y))^2$$

with domain $\mathcal{F} = C_c(V)_{\mathcal{E}^1}$. 

The “formal Laplacian” (Keller-Lenz):

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \in V} \omega(x, y)(u(x) - u(y)).$$
The graph metric is in general not adapted:

$$\frac{1}{\mu(x)} \sum_{y \in V} \omega(x, y) d_0^2(x, y) = \frac{1}{\mu(x)} \sum_{y \in V} \omega(x, y) = \text{Deg}(x).$$

Wojciechowski’s example of anti-tree: stochastic incompleteness with $r^{3+\varepsilon}$ type volume growth
Definition 3.2
A metric $d$ on a simple weighted graph $(V, \omega, \mu)$ is called adapted with jump size $c_0 > 0$ if

1. $\frac{1}{\mu(x)} \sum_{y \in V} \omega(x, y) d^2(x, y) \leq 1$, for each $x \in V$;
2. $\omega(x, y) = 0$ for $(x, y)$ with $d(x, y) > c_0$.

Example 3.3
Let $\sigma(x, y) = \min\left\{ \frac{1}{\sqrt{\text{Deg}(x)}}, \frac{1}{\sqrt{\text{Deg}(x)}}, c_0 \right\}$ for $x \sim y$. Define

$d_\sigma = \inf\left\{ \sum_{i=0}^{n-1} \sigma(x_i, x_{i+1}) : x_0 = x, x_n = y, x_i \sim x_{i+1}, \forall 0 \leq i \leq n - 1 \right\}$
Theorem 3.4 (Folz)

Let $(V, \omega, \mu)$ be a simple weighted graph. Let $d$ be an adapted metric such that all closed metric balls $B_d(x, r)$ are finite. If the volume growth with respect to $d$ satisfies:

$$
\int_{\infty}^{\infty} \frac{rdr}{\log (\mu(B_d(x_0, r)))} = \infty,
$$

for some reference point $x_0 \in V$, then the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is stochastically complete.

Folz’s strategy:

1. construct a related metric graph with loops;
2. compare the processes and volume growth;
3. reduction to Sturm’s theorem.
Sketch of an analytic proof

Construction of a metric graph $X$:

1. an orientation: $\tau : E \to \{1, -1\}$, satisfying $\tau((x, y)) = -\tau((y, x))$ for all $(x, y) \in E$, $E_+ := \tau^{-1}(\{1\})$;

2. positive weights: $\ell(e) = d(x, y)$, $p(e) = \omega(x, y)d(x, y)$ for $e = (x, y) \in E_+$;

3. marked intervals $\{I(e)\}_{e \in E_+}$, where $I(e) = [0, \ell(e)] \times \{e\}$.

Natural gluing:

$$\pi : \bigsqcup_{e \in E_+} I(e) \to X.$$
Quotient metric: \( d_\ell \) through \( \pi \);
Push-forward measure: \( \tilde{\mu} = \pi_* \left( \bigoplus_{e \in E_+} p(e) m(e) \right) \);
Dirichlet form:

\[
\tilde{\mathcal{E}}(u, u) = \sum_{e \in E_+} p(e) \int_0^{\ell(e)} (u'|_{l(e)})^2 dm(e),
\]

with domain: \( \tilde{\mathcal{F}} = \overline{C_{\text{Lip},c}^{\tilde{\mathcal{E}}_1}} \).
Key facts: \( d_\ell = \rho \geq d \)

\[
\tilde{\mu} \left( B^X_\rho(x_0, r) \right) \leq \mu \left( B^V_d(x_0, r) \right).
\]
Simplification: suffices to consider the case
\[ \mu(x) = \sum_y \omega(x, y) d^2(x, y) \] for each \( x \in V \).

Strategy:
- volume growth (♠) for the weighted graph
\[ \Rightarrow \] volume growth ($) for the metric graph
\[ \Rightarrow \] stochastic completeness of the metric graph
\[ ? \]
\[ \Rightarrow \] stochastic completeness of the weighted graph
Let $\{\tilde{u}_n\} \subset \tilde{F}$ be a sequence of functions satisfying

$$0 \leq \tilde{u}_n \leq 1, \quad \lim_{n \to \infty} \tilde{u}_n = 1 \quad \tilde{\mu}\text{-a.e.}$$

such that

$$\lim_{n \to \infty} \tilde{E}(\tilde{u}_n, \tilde{v}) = 0$$

holds for any $\tilde{v} \in \tilde{F} \cap L^1(\tilde{X}, \tilde{\mu})$. 

Define $u_n = \tilde{u}_n |_V$. For each $w \in \mathcal{F} \cap L^1(V, \mu)$, define $\tilde{w}$ on $X$ by linear interpolation. Formally,

$$E(u_n, w) = \sum_{e=(x,y) \in E_+} \omega(x, y) (u_n(x) - u_n(y)) (w(x) - w(y))$$

$$= \sum_{e=(x,y) \in E_+} \omega(x, y) d(x, y) \int_{l(e)} \tilde{u}_n'(t) \tilde{w}'(t) dt$$

$$= \tilde{E}(\tilde{u}_n, \tilde{w}) \to 0.$$

Checking that everything works rigorously only involves some elementary and fun calculations.
Claim 1: The sequence \( \{u_n\} \subset \mathcal{F} \).

By the recent result of H.-Keller-Masamune-Wojciechowski on essential self-adjointnees:

\[
\mathcal{F} = \mathcal{F}_{\text{max}} = \{ u : \sum_{x \in V} u^2(x) \mu(x) + \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \omega(x, y) (u(x) - u(y))^2 < \infty \}.
\]
For each $\tilde{u} \in \tilde{\mathcal{F}}$ with $u = \tilde{u}|_V$,

$$\mathcal{E}(u, u) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \omega(x, y) (u(x) - u(y))^2$$

$$= \sum_{e=(x,y) \in E_+} \omega(x, y) \left( \int_0^{l(e)} \tilde{u}'(t) dt \right)^2$$

$$\leq \sum_{e=(x,y) \in E_+} \omega(x, y) d(x, y) \left( \int_0^{l(e)} (\tilde{u}'(t))^2 dt \right)$$

$$= \tilde{\mathcal{E}}(\tilde{u}, \tilde{u}).$$
To show that \( u_n \in L^2(V, \mu) \):

\[
\left( \sup_{t \in [0, l]} |\tilde{u}(t)| \right)^2 \leq \coth(l) \int_0^l (\tilde{u}^2(t) + (\tilde{u}'(t))^2) \, dt.
\]

\[
\| \tilde{u} \|_{L^2(e)} \leq \frac{\coth(d(x, y))}{\omega(x, y) d(x, y)} \| \tilde{u} \|_{L^2(e)}^2 \frac{2}{W^{1,2}(I(e))}.
\]

Here

\[
\| \tilde{u} \|_{W^{1,2}(I(e))} \|_{L^2(e)}^2 = p(e) \int_0^{\ell(e)} (\tilde{u}^2(t) + (\tilde{u}'(t))^2) \, dt.
\]
\[
\sum_{x \in V} u^2(x)\mu(x) = \sum_{e=(x,y)\in E_+} \omega(x,y)d^2(x,y) \left( u^2(x) + u^2(y) \right)
\leq 2 \sum_{e=(x,y)\in E_+} d(x,y) \coth(d(x,y)) \| \tilde{u} \|_{W^{1,2}(I(e))}^2 
\leq C \mathcal{E}_1(\tilde{u}, \tilde{u}),
\]

where \( C = 2 \sup_{t \in (0, c_0]} t \coth(t) > 0. \)
Claim 2: For each $x \in V$, $\lim_{n \to \infty} u_n(x) = 1$.

Fix $x \in V$ and let $E_x \subseteq E_+$ be the set of marked intervals with a vertex being $x$.

Choose some $y_e \in (0, l(e))$ for each $e \in E_x$ such that $\lim_{n \to \infty} \tilde{u}_n(y_e) = 1$.

Define $\tilde{v}$: on $\bigcup_{e \in E_x} l(e)$ by $\tilde{v}(y) = \frac{1}{d(x, y_e)}(d(x, y_e) - d(x, y))_+$

for $y \in l(e)$ and extend it by 0 outside.

The function $\tilde{v}$ is compactly supported, Lipschitz and thus $\tilde{v} \in L^1(X, \tilde{\mu}) \cap \tilde{\mathcal{F}}$. 
Then we have that

\[
0 = \lim_{n \to \infty} \tilde{G}(\tilde{u}_n, \tilde{v}) = \lim_{n \to \infty} \sum_{e \in E_x} \omega(e) \ell(e) \int_{l(e)} \tilde{u}_n'(t)\tilde{v}'(t) dt \\
= \lim_{n \to \infty} \sum_{e \in E_x} \omega(e) \ell(e) \frac{1}{d(x, y_e)} (\tilde{u}_n(x) - \tilde{u}_n(y_e)),
\]

whence \( \lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} \tilde{u}_n(x) = 1. \)
Claim 3: The function $\tilde{w} \in L^1(X, \tilde{\mu}) \cap \tilde{\mathcal{F}}$.

Let $\{w_n\} \subset C_c(V)$ be a sequence converging to $w$ in the $\mathcal{C}_1$ norm. Let $\tilde{w}_n$ be the extension of $w_n$ by linear interpolation in the same way as $\tilde{w}$.

For each $e = (x, y) \in E_+$, we have

$$\omega(x, y) d(x, y) \int_{I(e)} (\tilde{w}'(t))^2 \, dt = \omega(x, y) (w(x) - w(y))^2.$$
And

\[ \omega(x, y) d(x, y) \int_{I(e)} \tilde{w}^2(t) dt \]

\[ = \frac{1}{3} \omega(x, y) d^2(x, y) \left( w^2(x) + w(x)w(y) + w^2(y) \right) \]

\[ \leq \frac{1}{2} \omega(x, y) d^2(x, y) \left( w^2(x) + w^2(y) \right), \]

whence \( \tilde{\mathcal{E}}_1(\tilde{w}, \tilde{w}) \leq \mathcal{E}_1(w, w). \)

The same estimate

\[ \tilde{\mathcal{E}}_1(\tilde{w} - \tilde{w}_n, \tilde{w} - \tilde{w}_n) \leq \mathcal{E}_1(w - w_n, w - w_n) \]

holds for each \( n. \)
To show that $\tilde{w} \in L^1(X, \tilde{\mu})$, we need another elementary calculation for each $e = (x, y) \in E_+$:

$$\omega(x, y)d(x, y)\int_{l(e)} |\tilde{w}(t)|dt = \frac{1}{2}\omega(x, y)d^2(x, y)\left(|w(x)| + |w(y)|\right),$$

by properties of linear functions. It follows that

$$\|\tilde{w}\|_{L^1(X, \tilde{\mu})} = \frac{1}{2} \|w\|_{L^1(V, \mu)}.$$
Thank you for your attention!