Stochastic completeness of jump processes and random walks

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Outline

A cluster of related objects

- (X, d): a separable metric space such that all metric balls
$$B(x, r) = \{y \in X : d(x, y) \leq r\}$$
are compact;
- μ: a Radon measure with full support on X;
- $(\mathscr{E}, \mathscr{F})$: a regular Dirichlet form (symmetric);
e.g. $\mathscr{F} = H^1(\mathbb{R}^n)$, $\mathscr{E}(u, v) = \int_X (\nabla u \cdot \nabla v)\, dm$
- Δ: nonnegative definite generator;
- $(P_t)_{t>0}$: Markovian semigroup;
- $(\mathcal{X}_t)_{t\geq 0}$: Hunt process.

Typical examples

Beurling-Deny: for $u \in \mathscr{F} \cap C_c(X)$

$$\mathscr{E}(u,u) = \mathscr{E}^{(c)}(u,u) + \int_{X\times X - diag} (u(x)-u(y))^2 \, J(dx,dy) + \int_X u(x)v(x)k(dx),$$

- Brownian motion on a manifold, diffusions on metric graphs, $\to$ strongly local Dirichlet forms;
- α-stable process on $\mathbb{R}^n$, random walks on weighted graphs, $\to$ jump type process on metric spaces;
- jump-diffusion processes.

Stochastic (in)completeness

Various points of view

1. process: infinite lifetime almost surely;
2. process: upper escape rate, "forefront";
3. semigroup or heat kernel: $P_t\mathbf{1} = \mathbf{1}$, $\int_X p_t(x, y)\mu(dy) = 1$;
4. heat equation: nonnegative bounded solutions to

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) + \Delta u(x, t) = 0, \\ u(\cdot, 0) \equiv 0; \end{cases} \tag{1.1}$$

Various points of view (continued)

5. "generator": nonnegative bounded solutions to $\Delta u + \lambda u = 0$, or $\Delta u + \lambda u \leq 0$;
6. "generator" (weak Omori-Yau): $\Delta u \leq -\alpha$ on $\Omega_\alpha = \{x \in X : u(x) > \sup u - \alpha\}$;
7. Dirichlet form: $\exists \{u_n\} \subset \mathscr{F}$ with $0 \leq u_n \leq 1$, $\lim_{n\to\infty} u_n = \mathbf{1}$, s.t. $\lim_{n\to\infty} \mathscr{E}(u_n, v) = 0$, $\forall v \in L^1 \cap \mathscr{F}$;
8. large scale geometry: "very negative" curvature $\Longrightarrow$ stochastic incompleteness;
9. large scale geometry: "not very large" volume growth $\Longrightarrow$ stochastic completeness.

Riemannian manifold case:

- Grigor'yan (through heat equation): geodesic metric d, Riemannian volume μ,

$$\int^{\infty} \frac{rdr}{\ln\left(\mu\left(B_d(x_0, r)\right)\right)} = \infty, \qquad (\diamondsuit)$$

  implies stochastic completeness;
- special case: $\mu\left(B_d(x_0, r)\right) \leq \exp(CR^2)$;

Riemannian manifold case (continued):

- Gaffney, Hsu, Karp-Li, Takeda, Davies, Pigola-Rigoli-Setti, Takegoshi through different approaches; ▸ approaches
- Hsu and Qin: upper rate function,

$$t = \int^{\phi(t)} \frac{rdr}{\ln\left(\mu\left(B_d(x_0, r)\right)\right) + \ln\ln r}; \qquad (\clubsuit)$$

Volume growth for stochastic completeness ($\diamondsuit$) and escape rate ($\clubsuit$): sharp for model manifolds.

Strongly local case (Sturm):

- "calculus" through energy measure: $d\Gamma(u,u) \approx |\nabla u|^2 d\mu$;
- intrinsic metric ρ to replace d:

$$\rho(x,y) = \sup\{u(x) - u(y) : u \in \mathscr{F}_{loc} \cap C(X),\\ d\Gamma(u,u) \le d\mu\}.$$

- Assumption: $(X,\rho) \simeq (X,d)$;
- stochastically complete if

$$\int^{\infty} \frac{rdr}{\ln\left(\mu\left(B_\rho(x_0,r)\right)\right)} = \infty, \qquad (\$)$$

Key feature: $d\Gamma(\rho(x,\cdot),\rho(x,\cdot)) \le d\mu$.

Jump process case

Adapted metrics (Masamune-Uemura):

$$\sup_{x\in X}\int_{X\setminus\{x\}}(1\wedge d^2(x,y))J(x,y)\mu(dy)=M<\infty. \qquad (\heartsuit)$$

Remark 2.1

related: Lévy process, Takeda, Frank-Lenz-Wingert

Example 2.2

$\alpha-$*stable processes* $(\alpha \in (0,2))$ *on* $\mathbb{R}^n$*:*

$$J(x,y)=\frac{c_{n,\alpha}}{|x-y|^{n+\alpha}},$$

Remark 2.3

$(\heartsuit)$ is an analogue to $|\nabla d(x,\cdot)|\leq 1$ in the manifold case.

Jump process case:

- Masamune-Uemura: for any $\varepsilon > 0$

$$e^{-\varepsilon d(x_0,x)} \in L^1(X,\mu),$$

- Grigor'yan-H.-Masamune:

$$\liminf_{r\to\infty} \frac{\log \mu\left(B_d(x_0,r)\right)}{r\log r} < \frac{1}{2};$$

Jump process case:

- Masamune-Uemura-Wang (jump-diffusion):

$$\liminf_{r\to\infty} \frac{\log \mu\left(B_d(x_0, r)\right)}{r \log r} < \infty;$$

- Shiozawa-Uemura, Shiozawa: more general coefficients, d not necessarily a metric but a reasonable “length”.

Basic strategy for jump process:

- truncation and stability, $c > 0$ (jump size): $J'(x,y) = J(x,y)\mathbf{1}_{d(x,y)\le c}$.
- Davies' method: stochastic completeness $\Leftrightarrow$

$$\lim_{n\to\infty} \langle f - P_t f, g_n \rangle = 0$$

  for any $f \in Lip_c(X)$, where $\{g_n\} \subset L^2 \cap L^\infty(X,\mu)$, $0 \le g_n \uparrow \mathbf{1}$.
- Davies' method: estimate

$$\langle u_t - f, g_n \rangle^2 = \left(\int_0^t \mathcal{E}(u_s, g_n) ds \right)^2$$

Basic difficulty: lack of a chain rule due to non-locality.

Example 2.4

Let $\psi(x) = \exp(\alpha d(x, x_0))$,

$$|\psi(x) - \psi(y)| \leq \alpha d(x, y) \exp(\alpha c)\psi(x).$$

The function $1/\psi$ *is expected to compensate the volume growth.*

Open problem: volume growth criterion ($\diamondsuit$) and escape rate ($\clubsuit$)?

Weighted graphs:

- (V, E): a simple graph;
- $\omega : V \times V \to [0, \infty)$ as jump kernel $J(dx, dy)$
  1. $\omega(x, y) = \omega(y, x)$ for all $x, y \in V$;
  2. $(x, y) \in E \Leftrightarrow \omega(x, y) > 0$;
- $\mu : V \to (0, \infty)$ as a Radon measure.

Example 3.1

1. *"normalized":* $\omega = \mathbf{1}_E$, $\mu = \deg$;
2. *"physical" (Weber, Wojciechowski):* $\omega = \mathbf{1}_E$, $\mu \equiv 1$.

The graph metric d_0:

$$d_0(x,y) = \inf\{n : \exists \text{ a path of length } n \text{ connecting } x, y\}.$$

The regular Dirichlet form:

$$\mathscr{E}(u,u) = \frac{1}{2}\sum_x \sum_y \omega(x,y)\,(u(x)-u(y))^2$$

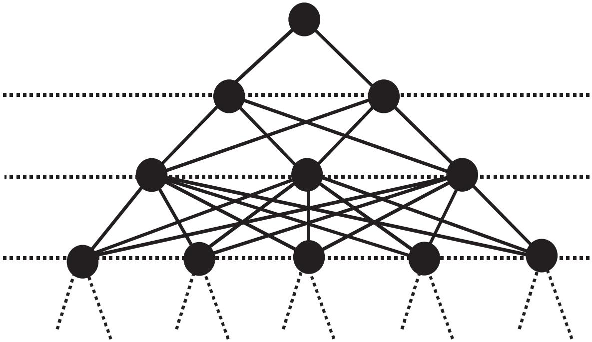
with domain $\mathscr{F} = \overline{C_c(V)}^{\mathscr{E}_1}$.
The "formal Laplacian" (Keller-Lenz):

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \in V} \omega(x,y)(u(x)-u(y)).$$

The graph metric is in general not adapted:

$$\frac{1}{\mu(x)}\sum_{y\in V}\omega(x,y)d_0^2(x,y)=\frac{1}{\mu(x)}\sum_{y\in V}\omega(x,y)=\mathrm{Deg}(x).$$

Wojciechowski's example of anti-tree: stochastic incompleteness with $r^{3+\varepsilon}$ type volume growth



Definition 3.2

A metric d on a simple weighted graph (V, ω, μ) is called adapted with jump size $c_0 > 0$ if

1. $\frac{1}{\mu(x)} \sum_{y \in V} \omega(x,y) d^2(x,y) \leq 1$, *for each* $x \in V$;
2. $\omega(x,y) = 0$ *for* (x,y) *with* $d(x,y) > c_0$.

Example 3.3

Let $\sigma(x,y) = \min\{\frac{1}{\sqrt{\mathrm{Deg}(x)}}, \frac{1}{\sqrt{\mathrm{Deg}(x)}}, c_0\}$ *for* $x \sim y$. *Define* $d_\sigma = \inf\{\sum_{i=0}^{n-1} \sigma(x_i, x_{i+1}) : x_0 = x, x_n = y, x_i \sim x_{i+1}, \forall 0 \leq i \leq n-1\}$

Theorem 3.4 (Folz)

Let (V, ω, μ) be a simple weighted graph. Let d be an adapted metric such that all closed metric balls $B_d(x, r)$ are finite. If the volume growth with respect to d satisfies:

$$\int^{\infty} \frac{rdr}{\log\left(\mu\left(B_d(x_0, r)\right)\right)} = \infty, \qquad (\spadesuit)$$

for some reference point $x_0 \in V$, then the corresponding Dirichlet form $(\mathscr{E}, \mathscr{F})$ is stochastically complete.

Folz's strategy:

1. construct a related metric graph with loops;
2. compare the processes and volume growth;
3. reduction to Sturm's theorem. ▸ Sturm

Sketch of an analytic proof

Construction of a metric graph X:

1. an orientation: $\tau : E \to \{1, -1\}$, satisfying $\tau((x, y)) = -\tau((y, x))$ for all $(x, y) \in E$, $E_+ := \tau^{-1}(\{1\})$;
2. positive weights: $\ell(e) = d(x, y)$, $p(e) = \omega(x, y)d(x, y)$ for $e = (x, y) \in E_+$;
3. marked intervals $\{I(e)\}_{e \in E_+}$, where $I(e) = [0, \ell(e)] \times \{e\}$.

Natural gluing:

$$\pi : \bigsqcup_{e \in E_+} I(e) \twoheadrightarrow X.$$

Quotient metric: d_ℓ through π;
Push-forward measure: $\tilde{\mu} = \pi_* \left(\oplus_{e \in E_+} p(e) m(e) \right)$;
Dirichlet form:

$$\tilde{\mathscr{E}}(u,u) = \sum_{e \in E_+} p(e) \int_0^{\ell(e)} (u'|_{I(e)})^2 dm(e),$$

with domain: $\tilde{\mathscr{F}} = \overline{C_{Lip,c}}^{\tilde{\mathscr{E}}_1}$.
Key facts: $d_\ell = \rho \geq d$

$$\tilde{\mu}\left(B_\rho^X(x_0,r)\right) \leq \mu\left(B_d^V(x_0,r)\right).$$

Simplification: suffices to consider the case
$\mu(x) = \sum_y \omega(x, y) d^2(x, y)$ for each $x \in V$.
Strategy:

volume growth (♠) for the weighted graph
$\Rightarrow$ volume growth ($) for the metric graph
$\Rightarrow$ stochastic completeness of the metric graph
$\overset{?}{\Rightarrow}$ stochastic completeness of the weighted graph

Let $\{\tilde{u}_n\} \subset \tilde{\mathscr{F}}$ be a sequence of functions satisfying

$$0 \leq \tilde{u}_n \leq 1, \lim_{n\to\infty} \tilde{u}_n = 1 \quad \tilde{\mu}\text{-a.e.}$$

such that

$$\lim_{n\to\infty} \tilde{\mathscr{E}}(\tilde{u}_n, \tilde{v}) = 0$$

holds for any $\tilde{v} \in \tilde{\mathscr{F}} \cap L^1(X, \tilde{\mu})$.

Define $u_n = \tilde{u}_n|_V$. For each $w \in \mathscr{F} \cap L^1(V, \mu)$, define $\tilde{w}$ on X by linear interpolation. Formally,

$$\begin{aligned}
\mathscr{E}(u_n, w) &= \sum_{e=(x,y)\in E_+} \omega(x,y)\,(u_n(x) - u_n(y))\,(w(x) - w(y)) \\
&= \sum_{e=(x,y)\in E_+} \omega(x,y) d(x,y) \int_{I(e)} \tilde{u}_n'(t)\tilde{w}'(t)dt \\
&= \tilde{\mathscr{E}}(\tilde{u}_n, \tilde{w}) \to 0.
\end{aligned}$$

Checking that everything works rigorously only involves some elementary and fun calculations.

Claim 1: The sequence $\{u_n\} \subset \mathscr{F}$.
By the recent result of H.-Keller-Masamune-Wojciechowski on essential self-adjointnees:

$$\begin{aligned}\mathscr{F} =& \mathscr{F}_{\max} \\ =& \{u : \sum_{x\in V} u^2(x)\mu(x) \\ & + \frac{1}{2}\sum_{x\in V}\sum_{y\in V}\omega(x,y)\left(u(x)-u(y)\right)^2 < \infty\}.\end{aligned}$$

For each $\tilde{u} \in \tilde{\mathscr{F}}$ with $u = \tilde{u}|_V$,

$$\begin{aligned}
\mathscr{E}(u,u) &= \frac{1}{2}\sum_{x\in V}\sum_{y\in V}\omega(x,y)\left(u(x)-u(y)\right)^2 \\
&= \sum_{e=(x,y)\in E_+}\omega(x,y)\left(\int_0^{l(e)}\tilde{u}'(t)dt\right)^2 \\
&\leq \sum_{e=(x,y)\in E_+}\omega(x,y)d(x,y)\left(\int_0^{l(e)}\left(\tilde{u}'(t)\right)^2 dt\right) \\
&= \tilde{\mathscr{E}}(\tilde{u},\tilde{u}).
\end{aligned}$$

To show that $u_n \in L^2(V,\mu)$:

$$\left(\sup_{t\in[0,l]} |\tilde{u}(t)|\right)^2 \leq \coth(l) \int_0^l \left(\tilde{u}^2(t) + (\tilde{u}'(t))^2\right) dt.$$

$$\| \tilde{u}|_{I(e)} \|_{\sup}^2 \leq \frac{\coth\left(d(x,y)\right)}{\omega(x,y)d(x,y)} \| \tilde{u}|_{I(e)} \|_{W^{1,2}(I(e))}^2 .$$

Here

$$\| \tilde{u}|_{I(e)} \|_{W^{1,2}(I(e))}^2 := p(e) \int_0^{\ell(e)} \left(\tilde{u}^2(t) + (\tilde{u}'(t))^2\right) dt.$$

$$
\begin{aligned}
\sum_{x\in V} u^2(x)\mu(x) &= \sum_{e=(x,y)\in E_+} \omega(x,y) d^2(x,y)\left(u^2(x)+u^2(y)\right) \\
&\leq 2 \sum_{e=(x,y)\in E_+} d(x,y)\coth\left(d(x,y)\right) \parallel \tilde{u} \parallel^2_{W^{1,2}(I(e))} \\
&\leq C\tilde{\mathscr{E}}_1(\tilde{u},\tilde{u}),
\end{aligned}
$$

where $C = 2\sup_{t\in(0,c_0]} t\coth(t) > 0$.

Claim 2: For each $x \in V$, $\lim_{n\to\infty} u_n(x) = 1$.
Fix $x \in V$ and let $E_x \subset E_+$ be the set of marked intervals with a vertex being x.
Choose some $y_e \in (0, l(e))$ for each $e \in E_x$ such that $\lim_{n\to\infty} \tilde{u}_n(y_e) = 1$.
Define $\tilde{v}$: on $\cup_{e\in E_x} I(e)$ by $\tilde{v}(y) = \frac{1}{d(x,y_e)}(d(x, y_e) - d(x, y))_+$ for $y \in I(e)$ and extend it by 0 outside.
The function $\tilde{v}$ is compactly supported, Lipschitz and thus $\tilde{v} \in L^1(X, \tilde{\mu}) \cap \tilde{\mathscr{F}}$.

Then we have that

$$0 = \lim_{n\to\infty} \tilde{\mathscr{E}}(\tilde{u}_n, \tilde{v}) = \lim_{n\to\infty} \sum_{e\in E_x} \omega(e)\ell(e) \int_{I(e)} \tilde{u}_n'(t)\tilde{v}'(t)dt$$

$$= \lim_{n\to\infty} \sum_{e\in E_x} \omega(e)\ell(e) \frac{1}{d(x, y_e)} (\tilde{u}_n(x) - \tilde{u}_n(y_e)),$$

whence $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \tilde{u}_n(x) = 1$.

Claim 3: The function $\tilde{w} \in L^1(X, \tilde{\mu}) \cap \tilde{\mathscr{F}}$.
Let $\{w_n\} \subset C_c(V)$ be a sequence converging to w in the $\mathscr{E}_1$ norm. Let $\tilde{w}_n$ be the extension of w_n by linear interpolation in the same way as $\tilde{w}$.
For each $e = (x, y) \in E_+$, we have

$$\omega(x, y)d(x, y) \int_{I(e)} (\tilde{w}'(t))^2 \, dt = \omega(x, y) \left(w(x) - w(y)\right)^2 .$$

And

$$\begin{aligned}&\omega(x,y)d(x,y)\int_{I(e)}\tilde{w}^2(t)dt\\=&\frac{1}{3}\omega(x,y)d^2(x,y)\left(w^2(x)+w(x)w(y)+w^2(y)\right)\\\leq&\frac{1}{2}\omega(x,y)d^2(x,y)\left(w^2(x)+w^2(y)\right),\end{aligned}$$

whence $\tilde{\mathscr{E}}_1(\tilde{w},\tilde{w})\leq\mathscr{E}_1(w,w)$.
The same estimate
$\tilde{\mathscr{E}}_1(\tilde{w}-\tilde{w}_n,\tilde{w}-\tilde{w}_n)\leq\mathscr{E}_1(w-w_n,w-w_n)$ holds for each n.

To show that $\tilde{w} \in L^1(X, \tilde{\mu})$, we need another elementary calculation for each $e = (x, y) \in E_+$:

$$\omega(x,y)d(x,y)\int_{I(e)} |\tilde{w}(t)|dt = \frac{1}{2}\omega(x,y)d^2(x,y)\left(|w(x)| + |w(y)|\right),$$

by properties of linear functions. It follows that $\| \tilde{w} \|_{L^1(X,\tilde{\mu})} = \frac{1}{2} \| w \|_{L^1(V,\mu)}$.

Thank you for your attention!