Annealed Brownian motion in a heavy tailed Poissonian potential

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Introduction

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$$u(t,x) = E_0\left[\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}, B_t = x\right]$$

$\longrightarrow$ localization of the diffusion particle

1. Setting

- $\left(\{B_t\}_{t\geq 0}, P_x\right)$: $\kappa\Delta$-Brownian motion on $\mathbb{R}^d$
- $\left(\omega = \sum_i \delta_{\omega_i}, \mathbb{P}\right)$: Poisson point process on $\mathbb{R}^d$ with unit intensity

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Potential

For a non-negative and integrable function v,

$$V_\omega(x) := \sum_i v(x - \omega_i).$$

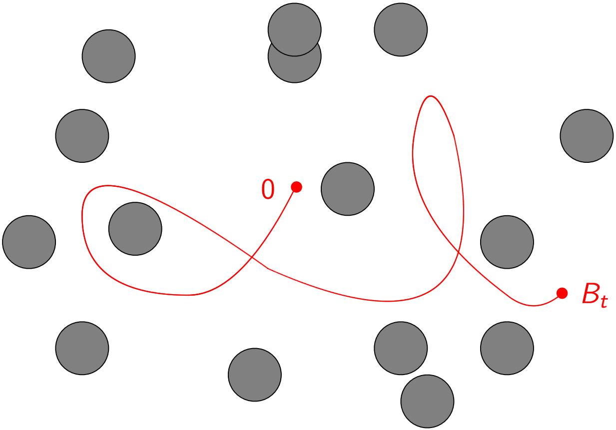
(Typically $v(x) = 1_{B(0,1)}(x)$ or $|x|^{-\alpha} \wedge 1$ with $\alpha > d$.)

Annealed measure

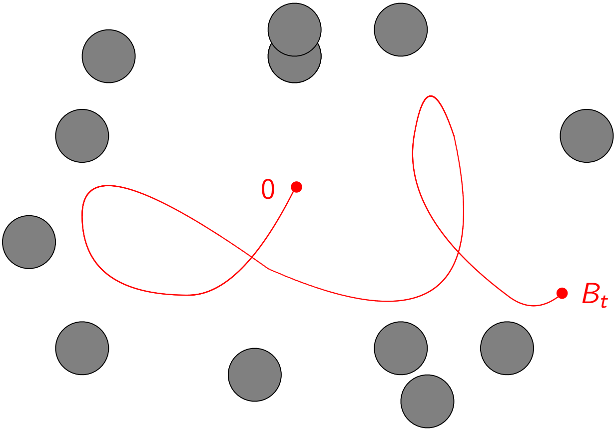
We are interested in the behavior of Brownian motion under the measure

$$Q_t(\,\cdot\,) = \frac{\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\mathbb{P}\otimes P_0(\,\cdot\,)}{\mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\right]}.$$

The configuration is not fixed and hence Brownian motion and ω_i's tend to avoid each other.



$\exp\{-\int_0^t V_\omega(B_s)\mathrm{d}s\}$: large, $\quad \mathbb{P}$: large, $\quad P_0$: small



$\exp\{-\int_0^t V_\omega(B_s)\mathrm{d}s\}$: large, $\mathbb{P}$: small, P_0 : large

2. Light tailed case

Donsker and Varadhan (1975)

When $v(x) = o(|x|^{-d-2})$ as $|x| \to \infty$,

$$\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(B_s)\,\mathrm{d}s \right\} \right] = \exp\left\{ -c(d,\kappa) t^{\frac{d}{d+2}} (1 + o(1)) \right\}$$

as $t \to \infty$.

Remark

$$c(d,\kappa) = \inf_U \{ \kappa \lambda^D(U) + |U| \}.$$

Suppose v is compactly supported for simplicity. Then

$$\begin{aligned} &\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(B_s)\, \mathrm{d}s \right\} \right] \\ &\quad \geq P_0 \left(B_{[0,t]} \subset U \right) \mathbb{P} \left(\omega(U) = 0 \right) \\ &\quad \approx \exp\{ -\kappa \lambda^D(U) t - |U| \}. \end{aligned}$$

Optimizing over U, we get the correct lower bound.

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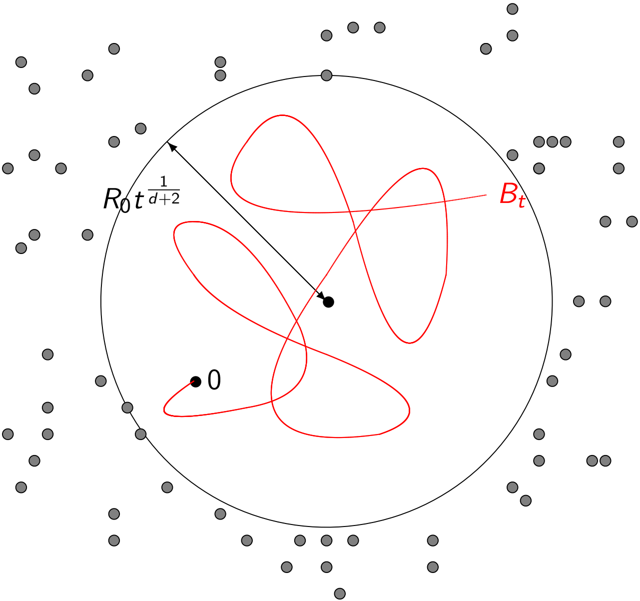
$$\text{maximizer} = B(x, t^{\frac{1}{d+2}} R_0)$$


$R_0 t^{\frac{1}{d+2}}$
B_t
0

One specific strategy gives dominant contribution to the partition function.

$$\Downarrow$$

It occurs with high probability under the annealed path measure.

Sznitman (1991, $d = 2$) and Povel (1999, $d \geq 3$)

When v has a compact support, there exists

$$D_t(\omega) \in B\big(0, t^{\frac{1}{d+2}}(R_0 + o(1))\big)$$

such that

$$Q_t\left(B_{[0,t]} \subset B\big(D_t(\omega), t^{\frac{1}{d+2}}(R_0 + o(1))\big)\right) \xrightarrow{t\to\infty} 1.$$

Remark

Bolthausen (1994) proved the corresponding result for two-dimensional random walk model.

3. Heavy tailed case

Pastur (1977)

When $v(x) \sim |x|^{-\alpha}$ ($\alpha \in (d, d+2)$) as $|x| \to \infty$,

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(B_s) \mathrm{d}s \right\} \right] = \exp \left\{ -a_1 t^{\frac{d}{\alpha}} (1 + o(1)) \right\},$$

where $a_1 = |B(0,1)| \Gamma\left(\frac{\alpha - d}{\alpha}\right)$.

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Unfortunately, this first order asymptotics tells us little about the Brownian motion.

In fact, Pastur's proof goes as follows:

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(B_s) \mathrm{d}s \right\} \right]$$
$$\approx \mathbb{E}[\exp\{-tV_\omega(0)\}]$$
$$\sim \exp\{-a_1 t^{\frac{d}{\alpha}}\}.$$

In fact, Pastur's proof goes as follows:

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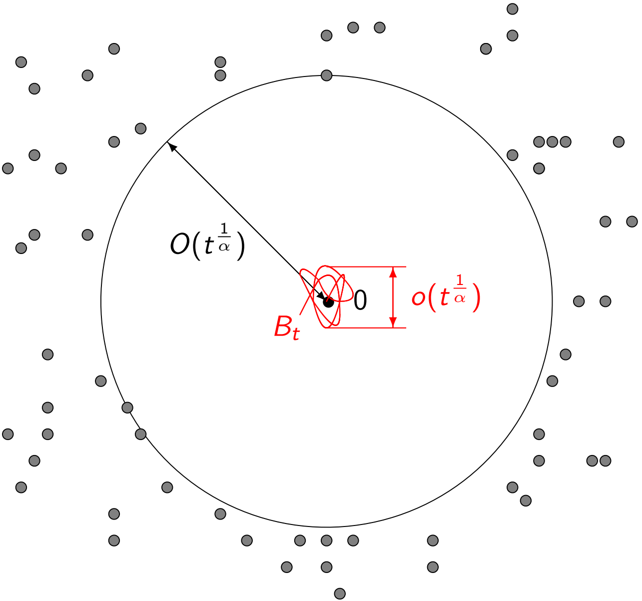
The effort of the Brownian motion is hidden in the higher order terms. A bit more careful inspection of the proof shows...


$O(t^{\frac{1}{\alpha}})$
B_t
0
$o(t^{\frac{1}{\alpha}})$

F. (2011)

When $v(x) = |x|^{-\alpha} \wedge 1$ $(d < \alpha < d+2)$,

$$\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(B_s) \mathrm{d}s \right\} \right] = \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - (a_2 + o(1)) t^{\frac{\alpha+d-2}{2\alpha}} \right\},$$

where

$$a_2 := \inf_{\|\phi\|_2 = 1} \left\{ \int \kappa |\nabla \phi(x)|^2 + C(d, \alpha) |x|^2 \phi(x)^2 \, \mathrm{d}x \right\}.$$

Remark

The proof is an application of the general machinery developed by Gärtner-König 2000.

Recalling the Donsker-Varadhan LDP

$$P_0\left(\frac{1}{t}\int_0^t \delta_{B_s}\mathrm{d}s \sim \phi^2(x)\mathrm{d}x\right) \approx \exp\left\{-t\int \kappa|\nabla\phi(x)|^2\mathrm{d}x\right\},$$

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In particular, since $P_0\left(B_{[0,t]} \subset B(x,R)\right) \approx \exp\{-tR^{-2}\}$, the localization scale should be

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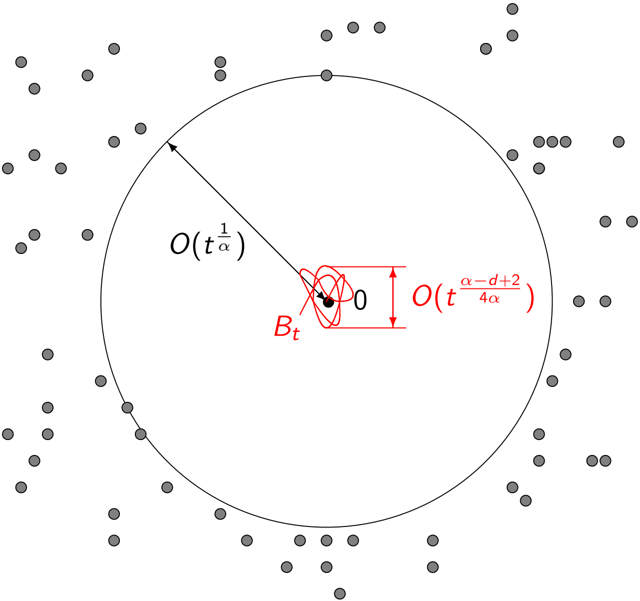
In addition, the term $\int C(d,\alpha)|x|^2\phi(x)^2\,\mathrm{d}x$ says that V_ω (locally) looks like a parabola.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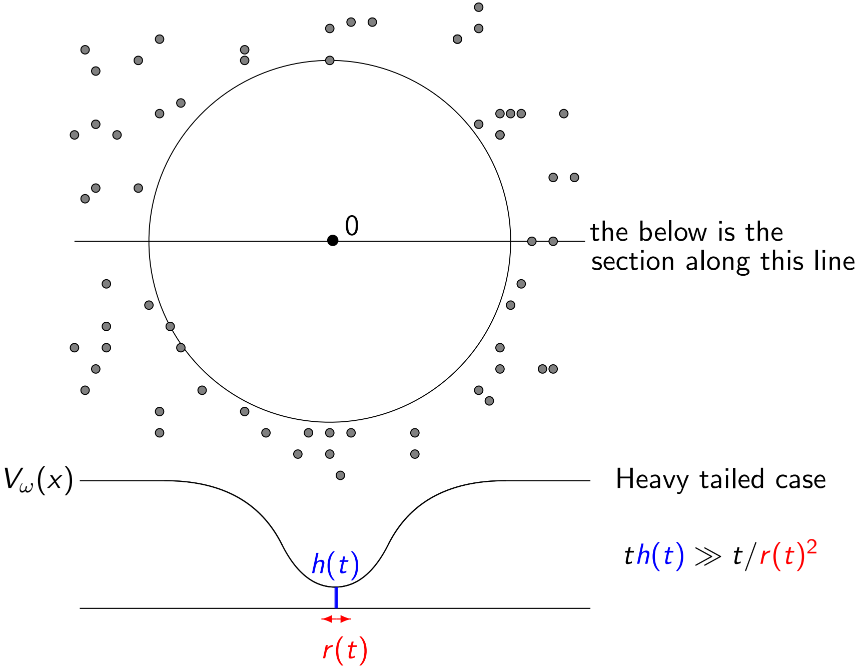
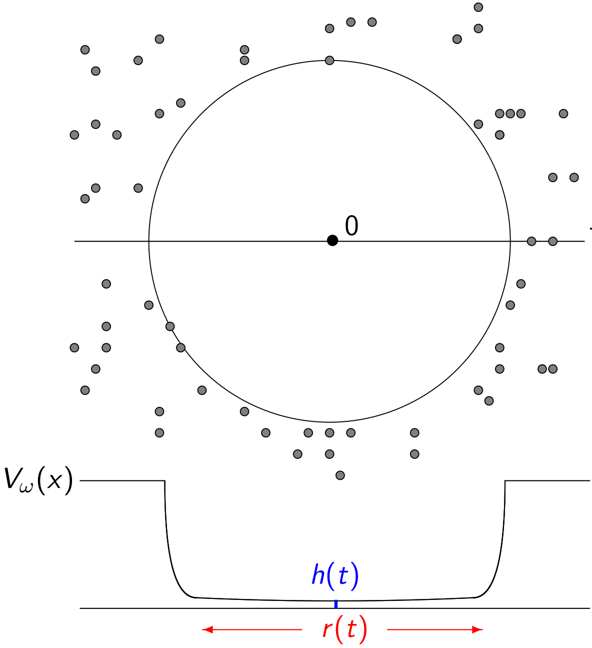

0
the below is the
section along this line
$V_\omega(x)$
Heavy tailed case
$t h(t) \gg t/r(t)^2$
$h(t)$
$r(t)$



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Light tailed case

$$th(t) \ll t/r(t)^2$$

Main Theorem (F. 2012)

$$Q_t\left(B_{[0,t]} \subset B\left(0, t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\epsilon}\right)\right) \xrightarrow{t\to\infty} 1,$$

$$Q_t\Big(V_\omega(x) - V_\omega(m_t(\omega)) \sim t^{-\frac{\alpha-d+2}{\alpha}} C(d,\alpha)|x - m_t(\omega)|^2$$
$$\text{in } B(0, t^{\frac{\alpha-d+2}{4\alpha}+\epsilon})\Big) \xrightarrow{t\to\infty} 1,$$

$$\left\{t^{-\frac{\alpha-d+2}{4\alpha}} B_{t^{\frac{\alpha-d+2}{2\alpha}}s}\right\}_{s\geq 0} \xrightarrow{\text{in law}} \text{OU-process with}$$
$$\text{“random center”},$$

where $m_t(\omega)$ is the minimizer of V_ω in $B(0, t^{\frac{\alpha-d+2}{4\alpha}}\log t)$.

4. Outline of the proof

An important feature of this model is that there are two scales in the “optimal strategy”.

- ω lives in the scale $t^{1/\alpha}$,
- $\{B_s\}_{s \le t}$ lives in the scale $o(t^{1/\alpha})$.

4. Outline of the proof

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This, for instace, prevents us from using the "compactification by projecting on a torus".

But on the other hand, this helps us since it allows us to treat ω and B_s "separately".

Suppose we have a crude estimate

$$\sup_{0 \le s \le t} |B_s| = o(t^{1/\alpha}).$$

Then $B_s \approx 0$ viewed from ω. Thus it is plausible that

$$Q_t(\mathrm{d}\omega) \sim \frac{\mathbb{E}[\exp\{-tV_\omega(0)\} : \mathrm{d}\omega]}{\mathbb{E}[\exp\{-tV_\omega(0)\}]}.$$

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The right hand side is nothing but the Poisson point process with intensity $e^{-t(|x|^{-\alpha} \wedge 1)} \mathrm{d}x$.

$\longrightarrow$ potential locally looks like parabola,

$\longrightarrow$ localization, weak convergence.

It remains to verify $\sup_{0 \le s \le t} |B_s| = o(t^{1/\alpha})$. This is a localization but weaker than the main theorem.
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No! What we need is crude control which can be established with bare hands.

In what follows, we assume V_ω takes its minimum value at 0 for simplicity.

4.1 Crude control on the potential

Lemma 1

$$Q_t\left(V_\omega(0) \in \frac{d}{\alpha} a_1 t^{-\frac{\alpha-d}{\alpha}} + t^{-\frac{3\alpha-3d+2}{4\alpha}}(-M_1, M_1)\right) \to 1.$$

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Idea

$$Z_t\approx\mathbb{E}[\exp\{-tV_\omega(0)\}]\begin{cases}=\exp\left\{-a_1t^{\frac{d}{\alpha}}\right\},\\ \approx\sup_{h>0}\left[e^{-th}\mathbb{P}(V_\omega(0)\approx h)\right].\end{cases}$$

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$\frac{d}{\alpha}a_1 t^{-\frac{\alpha-d}{\alpha}}=h(t)$ is the maximizer.

$$\begin{aligned}\Rightarrow&\mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}: V_\omega(0) \text{ is far from } h(t)\right]\\ &\le \mathbb{E}\left[\exp\{-tV_\omega(0)\}: V_\omega(0)\text{ is far from } h(t)\right]=o(Z_t).\end{aligned}$$

Lemma 2

$$Q_t\Big(V_\omega(0) + V_\omega(x) \geq 2h(t) + c_1 t^{-\frac{\alpha-d+2}{\alpha}} |x|^2 \text{ for } t^{\frac{\alpha-d+6}{8\alpha}} < |x| < M_2 t^{\frac{\alpha-d+6}{8\alpha}} \Big) \to 1.$$

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By Lemma 1,

$$\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\lesssim \exp\{-th(t)\}=\exp\left\{-\frac{d}{\alpha}a_1 t^{\frac{d}{\alpha}}\right\}$$

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Idea

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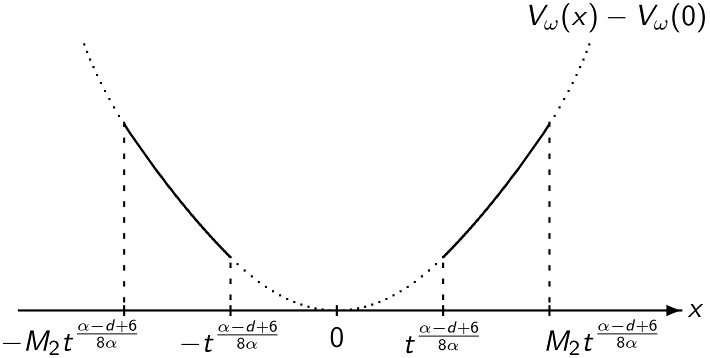
Then, use

$$\mathbb{E}\left[\exp\left\{ -\frac{t}{2}(V_\omega(0) + V_\omega(x)) \right\} \right] \approx \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - c_2 t^{\frac{d-2}{\alpha}} |x|^2 \right\}$$

and Chebyshev's inequality.

4.2 Crude control on the trajectory



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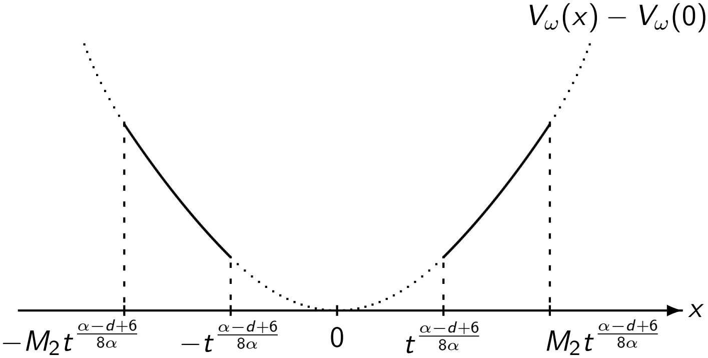

By "penalizing a crossing",

$$Q_t\left(B_{[0,t]} \subset B\left(0, M_2 t^{\frac{\alpha-d+6}{8\alpha}}\right)\right) \to 1.$$

How about the critical case $\alpha = d + 2$?

Ôkura (1981) proved

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ - \int_0^t V_\omega(B_s) \, \mathrm{d}s \right\} \right] = \exp \left\{ -\tilde{c}(d, \kappa) t^{\frac{d}{d+2}} (1 + o(1)) \right\}$$

as $t \to \infty$, where

$$\tilde{c}(d, \kappa) = \inf_{\|\phi\|_2 = 1} \left\{ \int \kappa |\nabla \phi(x)|^2 + 1 - \exp \left\{ - \int \frac{\phi(y)^2}{|x - y|^{d+2}} \, \mathrm{d}y \right\} \mathrm{d}x \right\}.$$

Thank you!

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