Brownian Motion with Darning applied to KL and BF equations for planar slit domains

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September 26, 2012
at Okayama University
Stochastic Analysis and Applications
German Japanese bilateral project

Objective of my talk

A domain of the form $D = \mathbb{H} \setminus \bigcup_{k=1}^{N} C_k$ is called a standard slit domain, where $\mathbb{H}$ is the upper half-plane and $\{C_k\}$ are mutually disjoint line segments parallel to x-axis contained in $\mathbb{H}$.

We fix a standard slit domain D and consider a Jordan arc

$$\gamma : [0, t_\gamma] \to \overline{D}, \quad \gamma(0) \in \partial\mathbb{H}, \quad \gamma(0, t_\gamma] \subset D. \tag{1.1}$$

For each $t \in [0, t_\gamma]$, let

$$g_t : D \setminus \gamma[0, t] \to D_t \tag{1.2}$$

be the unique conformal map from $D \setminus \gamma[0, t]$ onto a standard slit domain $D_t = \mathbb{H} \setminus \bigcup_{k=1}^{N} C_k(t)$ satisfying a hydrodynamic normalization

$$g_t(z) = z + \frac{a_t}{z} + o(1), \qquad z \to \infty. \tag{1.3}$$

a_t is called half-plane capacity and it can be shown to be a strictly increasing left-continuous function of t with $a_0 = 0$.

We write

$$\xi(t) = g_t(\gamma(t)) \ (\in \partial\mathbb{H}), \quad 0 \le t \le t_\gamma. \tag{1.4}$$

In
[BF08] On chordal and bilateral SLE in multiply connected domains, *Math. Z.* **258**(2008), 241-265

R.O. Bauer and R.M. Friedrich have derived a chordal Komatu-Loewner equation

$$\frac{\partial^- g_t(z)}{\partial a_t} = -\pi\Psi_t(g_t(z), \xi(t)), \quad g_0(z) = z \in D, \quad 0 < t \le t_\gamma, \tag{1.5}$$

where $\frac{\partial^- g_t(z)}{\partial a_t}$ denotes the left partial derivative with respect to a_t.

This is an extension of the radial Komatu-Loewner equation obtained first by Y. Komatu
[K50] On conformal slit mapping of multiply-conected domains, *Proc. Japan Acad.* **26**(1950), 26-31

and later by Bauer-Friedrich
[BF06] On radial stochastic Loewner evolution in multiply connected domains, *J. Funct. Anal.* **237**(2006), 565-588

The kernel $\Psi_t(z,\zeta),\ z \in D_t,\ \zeta \in \partial\mathbb{H}$, appearing in (1.5) is an analytic function of $z \in D_t$ whose imaginary part is constant on each slit $C_k(t)$ of the domain D_t.

It is explicitly expressed in terms of the classical Green function of the domain D_t.

However the following problems have not been solved
neither in the radial case [K50], [BF06] nor in the chordal case [BF08]:

Problem 1 (continuity). Is a_t continuous in t ?

Problem 2 (differentiablility). If a_t were continuous in t, the curve γ can be reparametrized in a way that $a_t = 2t,\ 0 \le t \le t_\gamma$.

Is $g_t(z)$ differentiable in t so that (1.5) can be converted to the genuine KL equation ?

$$\frac{d}{dt} g_t(z) = -2\pi\Psi_t(g_t(z), \xi(t)), \quad g_0(z) = z \in D, \quad 0 < t \le t_\gamma. \tag{1.6}$$

g_t can be extended as a homeomorphism between $\partial(D \setminus \gamma[0,t])$ and ∂D_t.
The slit C_k is homeomorphic with the image slit $C_k(t)$ by g_t for each $1 \leq k \leq N$.
Denote by $z_k(t)$, $z'_k(t)$ the left and right endpoints of $C_k(t)$.

[BF06], [BF08] went on further to make the following claims:

Claim 1. The endpoints are subjected to the Bauer-Friedrch equation

$$\frac{d}{dt} z_k(t) = -2\pi\Psi_t(z_k(t), \xi(t)), \quad \frac{d}{dt} z'_k(t) = -2\pi\Psi_t(z'_k(t), \xi(t)), \tag{1.7}$$

Claim 2. Conversely, given a continuous function $\xi(t)$ on the boundary $\partial\mathbb{H}$, the BF-equation (1.7) can be solved uniquely in $z_k(t)$, $z'_k(t)$,
and then the KL-equation (1.6) can be solved uniquely in $g_t(z)$.

We aim at answering Problems 1 and 2 affirmatively,
establishing the genuine KL-equation (1.6) with $\Psi_t(z, \zeta)$ being
the complex Poisson kernel of BMD on D_t
and moreover legitimating Claims 1 and 2 made by Bauer-Friedrich.

Known facts in simply connected case ($N = 0$)

- The continuity of a_t follows easily from the Carathéodory convegence theorem.
  The continuity of $\xi(t) \in \partial\mathbb{H}$ can also be shown by an complex analyitc argument.
- $D_t = \mathbb{H}$ and the complex Poisson kernel of $\mathbb{H}$ is given by

$$\Psi_t(z,\zeta) = \Psi(z,\zeta) = -\frac{1}{\pi}\frac{1}{(z-\zeta)}$$

  with

$$\Im\Psi(z,\zeta) = \frac{1}{\pi}\frac{1}{(x-\zeta)^2+y^2}, \quad z = x+iy,$$

  being the Poisson kernel of ABM on $\mathbb{H}$
  The equation (1.5) is reduced to the well known Loewner equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z)-\xi_t}, \quad g_0(z) = z, \tag{1.8}$$

  under the reparametrization $a_t = 2t$.
- Given a continuous motion $\xi(t)$ on $\partial\mathbb{H}$, $\{g_t\}$ and γ can be recovered by solving the Loewner equation (1.8).

- Given a probability measure on the collection of continuous curves γ on $\mathbb{H}$ connecting 0 and ∞
  that satisfies a domain Markov property and conformal invariance,
  the associated random motion $\xi(t)$ equals $\sqrt{\kappa}B_t$ for $\kappa > 0$ and the Brownian motion B_t.

- Conversely, given $\xi(t) = \sqrt{\kappa}B_t$ on $\partial\mathbb{H}$,
  the associated trace γ of the stochastic(Schramm) Loewner evolution (SLE) $\{g_t\}$
  behaves differently according to the parameter κ
  and is linked to scaling limits of certain random processes.

Definition of BMD

Let $D = \mathbb{H} \setminus \bigcup_{k=1}^{N} C_k$ be a standard slit domain.
A Brownian motion with darning (BMD) Z^* for D is, roughly speaking, a diffusion process on $\mathbb{H}$ absorbed at $\partial\mathbb{H}$ and reflected at each slit C_j but by regarding C_j as a single point c_k^*.
To be more precise, let

$$D^* = D \cup K^*, \quad K^* = \{c_1^*, c_2^*, \cdots, c_N^*\} \tag{2.1}$$

and define a negihborhood U_j^* of each point c_j* in D^* by $\{c_j^*\} \cup (U_j \setminus C_j)$ for any neighborhood U_j of C_j in $\mathbb{H}$.
Denote by m the Lebesgue measure on D and by m^* its zero extension to D^*.
Let $Z^0 = (Z_t^0, \mathbb{P}_z^0)$ be the absorbing Brownian motion(ABM) on D.
In
[CF] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Changes, and Boundary Theory*, Princeton University Press, 2012,
the BMD Z^* for D is characterized as a unique m^*-symmetric diffusion extension of Z^0 from D to D^* with no killing at K^*.

Let $(\mathcal{E}^*, \mathcal{F}^*)$ be the Dirichlet form of Z^* on $L^2(D^*; m^*) = L^2(D; m)$.
It is a regular strongly local Dirichlet form on $L^2(D^*; m^*)$ described as

$$\begin{cases} \mathcal{E}^*(u,v) = \frac{1}{2}\mathbf{D}(u,v), \quad u,v \in \mathcal{F}^*, \\ \mathcal{F}^* = \{u \in W_0^{1,2}(\mathbb{H}) : u \text{ is constant } \ \mathbf{D}\text{-q.e on each } \ C_j\}. \end{cases}$$

The uniquely associated diffusion on D^* is the BMD Z^*.

Let γ be an analytic Jordan curve surrounding a slit C_j, namely,
$\gamma \subset D, \ \text{ins}\gamma \supset C_j, \ \overline{\text{ins}\gamma} \cap C_k = \emptyset, \ k \neq j.$
For a harmonic function u defined in a neighborhood of C_j, the value

$$\int_\gamma \frac{\partial u(\zeta)}{\partial \mathbf{n}_\zeta} ds(\zeta)$$

is independent of the choice of such curve γ with the normal vector $\mathbf{n}$ pointing toward C_j and arc length s. This value is called the period of u around C_j.

Complex Poisson kernel of BMD

- Suppose v on D^* is harmonic with respect to the BMD Z^* on D^*. Then the period of v around C_j equals 0 for any $1 \le j \le N$.
  In particular, $-v\big|_D$ admits a harmonic conjugate u on D uniquely up to an additive real constant so that $f(z) = u(z) + iv(z),\ z \in D$, is an analytic function on D.
- We call the 0-order resolvent density function $G^*(x,y),\ x \in D^*,\ y \in D$, of the BMD Z^* the Green function of Z^*.
  From the zero period property of BMD-harmonic function, we can deduce

$$G^*(z,\zeta) = G(z,\zeta) + 2\Phi(z) \cdot \mathcal{A}^{-1} \cdot^t \Phi(\zeta), \quad z \in D^*,\ \zeta \in D. \tag{2.2}$$

Here $G(z,\zeta)$ is the Green function (0-order resolvent density) of the ABM Z^0 on D, $\Phi(z)$ is the N-vector with j component

$$\varphi^{(j)}(z) = \mathbb{P}^0_z(Z^0_{\zeta^0-} \in \partial A_j), \quad z \in D,\ 1 \le j \le N,$$

and $\mathcal{A}$ is an $N \times N$-matrix whose (i,j)-component p_{ij} equals the period of $\varphi^{(j)}$ around C_i, $1 \le i, j \le N$.

We now define the Poisson kernel of Z^* by
$K^*(z,\zeta) = -\frac{1}{2}\frac{\partial}{\partial \mathbf{n}_\zeta} G^*(z,\zeta),\ z \in D^*, \zeta \in \partial\mathbb{H}$, so that

$$K^*(z,\zeta) = -\frac{1}{2}\frac{\partial}{\partial \mathbf{n}_\zeta} G(z,\zeta) - \Phi(z)\cdot \mathcal{A}^{-1}\cdot^t \frac{\partial}{\partial \mathbf{n}_\zeta}\Phi(\zeta),\ z \in D^*,\ \zeta \in \partial\mathbb{H}. \quad (2.3)$$

$K^*(\cdot,\zeta),\ \zeta \in \partial\mathbb{H}$, is a Z^*-harmonic function of z on D^* for each $\zeta \in \partial\mathbb{H}$. Therefore there exists a function $\Psi(z,\zeta)$ analytic in $z \in D$ with imaginary part $K^*(z,\zeta)$ uniquely under the normalization condition

$$\lim_{z\to\infty} \Psi(z,\zeta) = 0. \quad (2.4)$$

$\Psi(z,\zeta),\ z \in D,\ \zeta \in \partial\mathbb{H}$, is called the complex Poisson kernel of the BMD Z^*.

We can give an alternative derivation of the KL-equation (1.5) in terms of the left derivative in a_t but with the BMD-complex Poisson kernel $\Psi_t(z,\zeta)$ of the slit domain D_t appearing on its right hand side.

Strategy

I A probabilistic representation of $\Im g_t(z)$ in terms of BMD

II continuity of $g_t(z)$ in t with some uniformity in z

III continuity of a_t

IV continuity of $\xi(t)$

V continuity of D_t

VI Lipschitz continuity of the BMD–complex Poisson kernel $\Psi(z, \zeta)$

I $\Longrightarrow$ II $\Longrightarrow$ III, IV, V
solving particularily Problem 1

VI combinded with II, IV, V
$\Longrightarrow$ continuity of $\Psi_t(g_t(z), \xi(t))$ (right hand side of (1.6))
$\Longrightarrow$ differentiability of $g_t(z)$
solving Problem 2

Probabilistic expression of $\Im g_t(z)$ and continuity of $g_t(z)$

We write $F_t = \gamma[0,t]$. Let
$Z^{\mathbb{H}} = (Z^{\mathbb{H}}_\cdot, \mathbb{P}^{\mathbb{H}}_z)$: the absorbing Brownian motion on $\mathbb{H}$
$Z^* = (Z^*_\cdot, \mathbb{P}^*_z)$: the BMD on $D^* = D \cup K^*$ with $K^* = \{c_1^*, \cdots, c_N^*\}$

For $r > 0$, let $\Gamma_r = \{z = x + iy : y = r\}$ and

$$v_t^*(z) := \lim_{r\to\infty} r \cdot \mathbb{P}^{\mathbb{H},*}_z(\sigma_{\Gamma_r} < \sigma_{F_t}), \quad z \in D^* \setminus F.$$

The function v_t^* is well defined by the above and Z^*-harmonic on $D^* \setminus F$.
Furthermore

$$v_t^*(z) = v_t(z) + \sum_{j=1}^N \mathbb{P}^{\mathbb{H}}_z\left(\sigma_K < \sigma_{F_t},\, Z^{\mathbb{H}}_{\sigma_K} \in C_j\right) v_t^*(c_j^*),\ z \in D \setminus F, \tag{3.1}$$

where

$$v_t(z) = \Im z - \mathbb{E}^{\mathbb{H}}_z[\Im Z^{\mathbb{H}}_{\sigma_{K\cup F_t}};\ \sigma_{K\cup F_t} < \infty]\ (\geq 0), \tag{3.2}$$

$$v_t^*(c_i^*) = \sum_{j=1}^{N} \frac{M_{t,ij}}{1 - R_{t,j}^*} \int_{\eta_j} v_t(z)\nu_j(dz), \quad 1 \le i \le N. \tag{3.3}$$

Here $\eta_1, \cdots, \eta_N$ are mutually disjoint smooth Jordan curve surrounding $C_1, \cdots, C_N$,

$$\nu_i(dz) = \mathbb{P}^*_{c_i^*}\left(Z^*_{\sigma_{\eta_i}} \in dz\right), \quad 1 \le i \le N, \tag{3.4}$$

$$R_{t,i}^* = \int_{\eta_i} \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{F_t},\ Z^{\mathbb{H}}_{\sigma_K} \in C_i\right)\nu_i(dz), \quad 1 \le i \le N, \tag{3.5}$$

and $M_{t,ij}$ is the (i,j)-entry of the matrix $M = \sum_{n=0}^{\infty}(Q_t^*)^n$ for a matrix Q_t^* with entries

$$q_{t,ij}^* = \begin{cases} \mathbb{P}^{\mathbb{H},*}_{c_i^*}\left(\sigma_{K^*} < \sigma_{F_t},\ Z^*_{\sigma_{K^*}} = c_j^*\right)/(1 - R_{t,i}^*) & \text{if } i \ne j, \\ 0 & \text{if } i = j, \end{cases} \quad 1 \le i,j \le N. \tag{3.6}$$

Moreover $v_t^*\big|_{D\setminus F}$ admits a unique harmonic conjugate u_t^* such that

$$g_t(z) = u_t^*(z) + iv_t^*(z), \quad z \in D \setminus F.$$

The way of constructing v^* in the above theorem is due to G.F. Lawler

[L06] The Laplacian-b random walk and the Schramm-Loewner evolution. *Illinois J. Math.* **50** (2006), 701-746

where the excursion reflected Brownian motion (ERBM) was utilized in place of the current BMD.

C_k^+ (resp. C_k^-) denotes the upper (resp. lower) side of the slit C_k, $1 \leq k \leq N$. We denote by $\partial_p C_k$ the set $C_k^+ \cup C_k^-$ with topology induced by the path distance in $\mathbb{H} \setminus C_k$. We also let $\partial_p K = \bigcup_{k=1}^N \partial_p C_k$.

From the preceding probabilistic expression, we can deduce the following:

For each fixed $s \in [0, t_\gamma]$,

$$\lim_{t \to s} g_t(z) = g_s(z),$$

uniformly in z on each compact subset of $D \cup \partial_p K \cup (\partial\mathbb{H} \setminus \{\gamma(0)\})$.

Lipschitz continuity of BMD-complex Poisson kernel

We denote by $\mathcal{D}$ the collection of 'labelled' standard slit domains.
For $D, \widetilde{D} \in \mathcal{D}$, define their distance $d(D, \widetilde{D})$ by

$$d(D, \widetilde{D}) = \max_{1 \le i \le N} (|z_i - \widetilde{z}_i| + |z_i' - \widetilde{z}_i'|), \tag{3.7}$$

where, for $D = \mathbb{H} \setminus \{C_1, C_2, \cdots, C_N\}$, z_i (resp. z_i') denotes the left (resp. right) end point of C_i, $1 \le i \le N$. $\widetilde{z}_i$, $\widetilde{z}_i'$, $1 \le i \le N$, are the corresponding points to $\widetilde{D}$.
$\{D_t : 0 \le t \le t_\gamma\}$ is a one parameter subfamily of $\mathcal{D}$.

For each $D \in \mathcal{D}$, the associated BMD-complex Poisson kernel $\Psi(z, \zeta)$, $z \in D$, $\zeta \in \partial\mathbb{H}$, is well defined.

The correspondence $\mathcal{D} \mapsto \Psi(z, \zeta)$ is Lipschitz continuous in the sense described in the next slide:

Let U_j, V_j, $1 \le j \le N$, be any relatively compact open subsets of $\mathbb{H}$ with

$$\overline{U}_j \subset V_j \subset \overline{V}_j \subset \mathbb{H},\ 1 \le j \le N,\ \overline{V}_j \cap \overline{V}_k = \emptyset,\ j \ne k.$$

We fix any $a > 0$ and $b > 0$ for which the subcollection $\mathcal{D}_0$ of $\mathcal{D}$ defined by

$$\mathcal{D}_0 = \{\mathbb{H} \setminus \cup_{j=1}^N C_j \in \mathcal{D} : C_j \subset U_j,\ |z_j - z_j'| > a,\ \mathrm{dist}(C_j, \partial U_j) > b,\ 1 \le j \le N\}$$

is non-empty. There exists then $\epsilon_0 > 0$ such that,
for any $\epsilon \in (0, \epsilon_0)$ and for any $D \in \mathcal{D}_0$ and $\widetilde{D} \in \mathcal{D}$ with $d(D, \widetilde{D}) < \epsilon$,
there exists a diffeomorphism $\widetilde{f}_\epsilon$ from $\mathbb{H}$ onto $\mathbb{H}$ satisfying

- $\widetilde{f}_\epsilon$ is sending D onto $\widetilde{D}$, linear on $\bigcup_{j=1}^N U_j$ and the identity map on $\mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j$,
- for some positive constant L_1 independent of $\epsilon \in (0, \epsilon_0)$ and of $D \in \mathcal{D}_0$,

$$|z - \widetilde{f}_\epsilon(z)| < L_1 \cdot \epsilon, \quad z \in \mathbb{H},$$

- for any compact subset Q of $\overline{\mathbb{H}}$ containing $\bigcup_{j=1}^N U_j$, and for any compact subset J of $\partial\mathbb{H}$,

$$|\Psi(z, \zeta) - \widetilde{\Psi}(\widetilde{f}_\epsilon(z), \zeta)| < L_{Q,J} \cdot \epsilon, \quad z \in (Q \setminus K) \cup \partial_p K,\ \zeta \in J,$$

where $\widetilde{\Psi}$ denotes the BMD-complex Poisson kernel for $\widetilde{D}$ and $L_{Q,J}$ is a positive constant independent of $\epsilon \in (0, \epsilon_0)$ and of $D \in \mathcal{D}_0$.

As we have seen in §2.2, the complex Poisson kernel $\Psi(z,\zeta)$ for a standard slit domain D can be obtained from the Green function $G(z,w)$ of D by repeating

to take normal derivatives at $\partial\mathbb{H}$,
to take the periods around the slits
to take line integrals of normal derivatives along smooth curves. The Lipschitz

continuity of $\Psi(z,\zeta)$ can be proved using the two perturbation formulae holding for the Green function $G(z,w)$ and the transformed one
$g(z,w,\epsilon)=\widetilde{G}(\widetilde{f}_\epsilon(z),\widetilde{f}_\epsilon(w))$:
We let $F=\bigcup_{i=1}^N(\overline{V}_i\setminus U_i)$. It holds that for any $\zeta\in\overline{\mathbb{H}}\setminus F$ and $w\in\overline{\mathbb{H}}$

$$g(\zeta,w,\epsilon)-G(\zeta,w)=\epsilon\int_F B_z^{(\epsilon)}G(z,\zeta)g(z,w,\epsilon)dx_1dx_2,\ \ z=x_1+ix_2,$$

$$g(\zeta,w,\epsilon)-G(\zeta,w)=\epsilon\int_F B_z^{(\epsilon)}G(z,\zeta)(G(z,w)+\epsilon\eta^{(\epsilon)}(z,w))dx_1dx_2,$$

where $\eta^{(\epsilon)}$ is a continuous function on $\overline{\mathbb{H}}\times\overline{\mathbb{H}}$ bounded there uniformly in $\epsilon>0$ and in $D\in\mathcal{D}_0$. It is important that domain of integration is restricted to F. These formulae can be shown following an iterior variation method in Section 15.1 of P.R. Garabedian

[G64] *Partial Differential Equation*, AMS Chelsia, 2007, republication of 1964 edition

Bauer-Friedrich equation of slit motions

For $t \in [0, t_\gamma]$, g_t maps $D = \mathbb{H} \setminus \bigcup_{j=1}^N C_j$ conformally onto $D_t = \mathbb{H} \setminus \bigcup_{j=1}^N C_j(t)$. For each $1 \le j \le N$, the endpoints $z_j(t),\ z_j'(t)$ of $C_j(t)$ satisfy the BF equation

$$\begin{cases} \frac{d}{dt}\Im z_j(t) = -2\pi \Im \Psi_t(z_j(t), \xi(t)), \\ \frac{d}{dt}\Re z_j(t) = -2\pi \Re \Psi_t(z_j(t), \xi(t)), \\ \frac{d}{dt}\Re z_j'(t) = -2\pi \Re \Psi_t(z_j'(t), \xi(t)), \end{cases} \tag{4.1}$$

To verify this, observe that there exist unique points

$$\widetilde{z}_j(t) \in \partial_p C_j, \text{ such that } g_t(\widetilde{z}_j(t)) = z_j(t),$$

$$\widetilde{z}_j'(t) \in \partial_p C_j, \text{ such that } g_t(\widetilde{z}_j'(t)) = z_j'(t)$$

as g_t is a homeomorphism between $\partial_p C_j$ and $\partial_p C_j(t)$, We denote the left and right end points of C_j by $z_j = a + ic,\ z_j' = b + ic$, respectively. We consider the open rectangles

$$R_+ = \{z : a < x < b,\ c < y < c + \delta\},\ R_- = \{z : a < x < b,\ c - \delta < y < c\},$$

and $R = R_+ \cup C_i \cup R_-$ for $\delta > 0$ with $R_+ \cup R_- \subset D \setminus \gamma[0, t_\gamma]$.
Since $\Im g_t(z)$ takes a constant value at C_j, g_t can be extended to an analytic function g_t^+ from R_+ to R across C_j by the Schwarz reflection.
Combining the preceding results with the Cauchy integral formula for the derivative of $g_t^+(z)$ in z,
$\frac{d}{dz} g_t^+(z)$ is shown to be C^1-function in $(t, z) \in (0, t_\gamma) \times R$.

Assume that $\widetilde{z}_j(t) \in C_j^+ \setminus \{z_j, z_j'\}$.
Since $g_t^+(\widetilde{z}_j(t))$ is the endpoint $z_j(t)$ of $C_j(t)$,
it can be shown that $\widetilde{z}_j(t)$ is a zero of $g_t^+(z) - g_t^+(\widetilde{z}_j(t))$ of order 2.
An informal differentiation of $\Re z_t(t) = \Re g_t^+(\widetilde{z}_j(t))$ in t then yields

$$\frac{d}{dt} \Re z_j(t) = \Re \frac{\partial}{\partial t} g_t(z)\big|_{z=\widetilde{z}_j(t)} + \Re[[g_t^+]'(\widetilde{z}_j(t)) \frac{d}{dt} \widetilde{z}_j(t)] = -2\pi \Re \Psi_t(z_j(t), \xi(t)).$$

When $\widetilde{z}_j(t) \in \partial_p C_j \cap B(z_j, \epsilon)$ for $0 < \epsilon < \frac{b-a}{2}$,
we map the region $B(z_j, \epsilon) \setminus \partial_p C_j$ onto $B(0, \sqrt{\epsilon}) \cap \mathbb{H}$ by

$$\psi(z) = (z - z_j)^{1/2}$$

and extend $f_t = g_t \circ \psi^{-1}$ analytically onto $B(0, \sqrt{\epsilon})$ by Schwarz reflection again.

We can make the same argument as above working with $(f_t, B(0, \sqrt{\epsilon}))$
in place of (g_t^+, R).

Solving BF equation for a given continuous $\xi(t)$

Define an open set $S \subset \mathbb{R}^{3N}$ by

$$S = \{(\mathbf{y}, \mathbf{x}, \mathbf{x}') \in \mathbb{R}^{3N} : \mathbf{y} > \mathbf{0},\ \mathbf{x} < \mathbf{x}', \\ \text{either } x'_j < x_k \text{ or } x'_k < x_j \text{ whenever } y_j = y_k,\ j \neq k\}. \tag{5.1}$$

The generic point of S is denoted by $\mathbf{s} = (\mathbf{y}, \mathbf{x}, \mathbf{x}'),\ \mathbf{y},\ \mathbf{x}\ , \mathbf{x}' \in \mathbb{R}^N$.
$\mathbf{s} \in S$ uniquely determines $D = D(\mathbf{s}) \in \mathcal{D}$ possessing slits C_j with endpoints $z_j = x_j + iy_j,\ z'_j = x'_j + iy_j$, and vice versa. The complex Poisson kernel $\Psi(z, \zeta)$ of BMD on $D \in \mathcal{D}$ will be designated by $\Psi_{\mathbf{s}}(z, \zeta)$ in terms of the corresponding point $\mathbf{s} \in S$. $|\mathbf{s}|$ will denote the Euclidean norm of $\mathbf{s} \in S$.

We fix a continuous real function $\xi(t),\ t \in [0, \infty)$.
We then define a vector field $\mathbf{f}(t, \mathbf{s}) = (f_k(t, \mathbf{s}))_{1 \leq k \leq 3N}$ on $[0, \infty) \times S$ by

$$\begin{cases} f_k(t, \mathbf{s}) = -2\pi \Im \Psi_{\mathbf{s}}(z_k, \xi(t)), & 1 \leq k \leq N, \\ f_{N+k}(t, \mathbf{s}) = -2\pi \Re \Psi_{\mathbf{s}}(z_k, \xi(t)), & 1 \leq k \leq N, \\ f_{2N+k}(t, \mathbf{s}) = -2\pi \Re \Psi_{\mathbf{s}}(z'_k, \xi(t)), & 1 \leq k \leq N. \end{cases}$$

We can then readily prove the claim described in the next slide:

(i) $\mathbf{f}(t, \mathbf{s})$ it is jointly continuous in $(t, \mathbf{s}) \in [0, \infty) \times S$.
(ii) $\mathbf{f}(t, \mathbf{s})$ is locally Lipschitz continuous in the following sense:
for any $\mathbf{s}_0 \in S$ and $0 < T < \infty$, there exists a neighborhood $U(\mathbf{s}_0) \subset S$ and a constant $L > 0$ such that

$$|\mathbf{f}(t, \mathbf{s}_1) - \mathbf{f}(t, \mathbf{s}_2)| \leq L|\mathbf{s}_1 - \mathbf{s}_2|, \quad \text{for any } \mathbf{s}_1,\ \mathbf{s}_2 \in U(\mathbf{s}_0),\ t \in [0, T].$$

(iii) For each $\tau \in [0, \infty)$, $\mathbf{s}_0 \in S$, the Cauchy problem

$$\frac{d}{dt}\mathbf{s}(t) = \mathbf{f}(t, \mathbf{s}(t)), \qquad \mathbf{s}(\tau) = \mathbf{s}_0, \tag{5.2}$$

has a unique solution in a neighborhood of $(\tau, \mathbf{s}_0)$ in $[0, \infty) \times S$.

$\mathbf{f}(t, \mathbf{s})$ is continuous in t for a fixed $\mathbf{s} \in S$ so that (i) follows from (ii).
(ii) is a special case of the result of 3.3. (iii) follows from (i) and (ii).

Let $\mathbf{s}(t) = (\mathbf{y}(t), \mathbf{x}(t), \mathbf{x}'(t))$ be the solution of (5.2).
Then (5.2) is reduced to the BF equation (4.1) with $\Psi_{\mathbf{s}(t)}$ in place of Ψ_t.

Solving KL equation for a given continuous $\xi(t)$

Suppose we are given a continuous real function $\xi(t), 0 \leq t < \infty$.
Let $\mathbf{s}(t); 0 \leq t < t_\xi$, be the solution of (5.2) with initial condition $\mathbf{s}(0) = \mathbf{s}_0$.
We write $D_t = D(\mathbf{s}(t)) \in \mathcal{D},\ t \in [0, t_\xi)$, and define $G \subset [0, t_\xi) \times \mathbb{H}$ by
$G = \cup_{t \in [0, t_\zeta)} \{t\} \times D_t$. G is a domain of $[0, t_\xi) \times \mathbb{H}$ because $t \mapsto D_t$ is continuous.

We then consider the Cauchy problem for the Komatu-Loewner equation:

$$\frac{d}{dt} g_t(z) = -2\pi \Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)), \quad g_0(z) = z \in D_0. \tag{5.3}$$

The next statements then hold true:
(i) $\Psi_{\mathbf{s}(t)}(z, \xi(t))$ is jointly continuous in $(t, z) \in G$.
(ii) $\Psi_{\mathbf{s}(t)}(z, \xi(t)),\ (t, z) \in G$, is locally Lipschitz continuous in the following sense:
for any $(t_0, z_0) \in G$, there exist $r > 0,\ \rho > 0,\ L > 0$, such that
$V = [(t_0 - r) \vee 0, t_0 + r] \times \{z : |z - z_0| \leq \rho\} \subset G$ and

$$|\Psi_{\mathbf{s}(t)}(z_1, \xi(t)) - \Psi_{\mathbf{s}(t)}(z_2, \xi(t))| \leq L \cdot |z_1 - z_2|, \quad \text{for any } (t, z_1),\ (t, z_2) \in V.$$

(iii) There exists a unique local solution of (5.3).

(i) can be shown by a similar argument to §3.1.
using the continuity of $t \mapsto D_t = D(\mathbf{s}(t))$ for a given continuous $\xi(t)$
in place of the continuity of $t \mapsto D_t$ for a given Jordan arc $\gamma(t)$.

The derivative in z of $\Psi_{\mathbf{s}(t)}(z, \xi(t))$ is also jointly continuous by virtue of (i) and the Cauchy integral formula. Therefore we readily get (ii).

(iii) follows from (i) and (ii).

In the above, we may take any set $\{\mathbf{s}(t) \in S : t \in [0, t_1)\}$ of points in S that is continuous in t in place of a solution of the BF equation (5.2).

In particular, we can conclude that $\{g_t(z) : t \in [0, t_\gamma]\}$ is the unique solution of the KL equation (1.6).

Basic properties of $(\mathbf{s}(t), \xi(t))$ for a given random curves γ

Let $\mathcal{D}$ be the collection of labeled standard slit domains and $S \subset \mathbb{R}^{3N}$ be the slit space defined by (5.1). $D \in \mathcal{D}$ and $\mathbf{s} \in S$ correspond each other in one-to-one way. A set $F \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if $\overline{F}$ is a compact continuium, $F = \overline{F} \cap \mathbb{H}$ and $\mathbb{H} \setminus F$ is simply connected. We let

$$\widehat{\mathcal{D}} = \{D = D' \setminus F : D' \in \mathcal{D},\ F \text{ compact } \mathbb{H}\text{−hull } F \cap \mathbb{H} \subset D'\}.$$

For $D \in \widehat{\mathcal{D}}$, let

$$W(D) = \{\gamma = \gamma(t), 0 \le t < \infty : \text{ Jordan curve } \gamma[0, \infty) \subset D\},$$

$$\mathcal{F}_t(D) = \sigma\{\gamma(s) : 0 \le s \le t\},\ \mathcal{F}(D) = \sigma\{\gamma(s) : 0 \le s < \infty\}.$$

Define a shift operator $\theta_t :\ W(D) \mapsto W(D)$ by $(\theta_t\gamma)(s) = \gamma(t + s),\ s \ge 0$.
For $D \in \widetilde{\mathcal{D}},\ z \in \overline{\mathbb{H}}$, consider a probability measure $\mathbb{P}_{D,z}$ on $(W(D), \mathcal{F}(D))$ satisfying

$$\mathbb{P}_{D,z}(\gamma(0) = z) = 1,$$

as well as DMP and CI described below:

DMP(domain Markov property): for any $t \geq 0$ and $A \subset \mathcal{F}(D \setminus \gamma[0,t])$

$$\mathbb{P}_{D,z}(\theta^{-1}\Lambda \big| \mathcal{F}_t) = \mathbb{P}_{D\setminus\gamma[0,t],\gamma(t)}(\Lambda),\ D \in \mathcal{D},\ z \in \partial\mathbb{H}.$$

CI (conformal invariance):
for any conformal map f from $D \in \widehat{\mathcal{D}}$ onto $f(D) \in \widehat{\mathcal{D}}$,

$$\mathbb{P}_{f(D),f(z)} = f_* \cdot \mathbb{P}_{D,z}, \quad D \in \mathcal{D}, \quad z \in D.$$

Consider a random motion $\mathbf{X}_t = (\mathbf{s}(t), \xi(t)) (\in S \times \mathbb{R}),\ t \geq 0,$
induced by the random curves γ.
For $\mathbf{s} \in S$ and $\xi \in \mathbb{R}$, define $\mathbb{P}_{(\mathbf{s},\xi)}$ by $\mathbb{P}_{(\mathbf{s},\xi)} = \mathbb{P}_{D(\mathbf{s}),(\xi,0)}$.

Then $(\mathbf{X}_t, \mathbb{P}_{(\mathbf{s},\xi)})$ enjoys the time homogeneous Markov property :
$\{\mathbf{X}_t\}_{t\geq 0}$ is $\mathcal{F}_t$-adapted and

$$\mathbb{P}_{(\mathbf{s},\xi)}(\mathbf{X}_{t+s} \in B \big| \mathcal{F}_t) = \mathbb{P}_{\mathbf{X}_t}(\mathbf{X}_s \in B), \quad t, s \geq 0,\ B \in \mathcal{B}(S \times \mathbb{R}).$$

Brownian scaling property of $(\mathbf{X}_t, \mathbb{P}_{(\mathbf{s},\xi)})$

For a curve γ on D with different parametrizations should be considered as different elements of $W(D)$,
We call $\gamma \in W(D)$ is of half-plane capacity parameter if $a_t = 2t,\ t \geq 0$.
$\widetilde{W}(D)$ denotes the collection of such elements in $W(D)$.
We assume that

$$\mathbb{P}_{(\mathbf{s},\xi)}(\widetilde{W}(D)) = 1, \quad \text{for } \mathbf{s} = \mathbf{s}(D), \text{ and for any } \xi \in \mathbb{R}. \tag{6.1}$$

In addition to (6.1), we assume that $\mathbb{P}_{(\mathbf{s},\xi)}$ satisfies (CI)
with respect to the dilation f: for a constant $c > 0$

$$f(z) = cz, \quad z \in D.$$

Then $(\mathbf{X}_t, \mathbb{P}_{(\mathbf{s},\xi)})$ enjoys the Brownian scaling property:for any $\xi \in \mathbb{R}$,

$$\{\frac{1}{c}\mathbf{X}_{c^2 t},\ t \geq 0\} \text{ under } \mathbb{P}_{(c\mathbf{s},c\xi)} \ \sim \ \{\mathbf{X}_t,\ t \geq 0\} \text{ under } \mathbb{P}_{(\mathbf{s},\xi)}.$$

THANK YOU FOR YOUR ATTENTION