Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians

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Introduction

- Spatially cut-off $P(\phi)_2$-Hamiltonian $-L + V_\lambda$ is a self-adjoint operator on $L^2(\mathcal{S}'(\mathbb{R}), d\mu)$, where $\lambda = 1/\hbar$.
- Formally:
  - $$d\mu(w) = \frac{1}{Z}\exp\left(-\frac{1}{2}\left(\sqrt{m^2-\Delta}w, w\right)_{L^2(\mathbb{R},dx)}\right)dw$$
  - $-L + V_\lambda$ is unitarily equivalent to
  $$-\Delta_{L^2(\mathbb{R})} + \lambda U(w/\sqrt{\lambda}) - \frac{1}{2}\mathrm{tr}(m^2-\Delta)^{1/2}$$
  on $L^2(L^2(\mathbb{R}, dx), dw)$

Introduction

where

$$
\begin{aligned}
U(w) &= \frac{1}{4}\int_{\mathbb{R}} (w'(x)^2 + m^2 w(x)^2)dx + V(w), \\
V(w) &= \int_{\mathbb{R}} :P(w(x)): g(x)dx,
\end{aligned}
$$

where P is a polynomial bounded below. It is natural to expect that there exists some relations between

- Asymptotic behavior of low-lying spectrum of the operator $-L + V_\lambda$ as $\lambda \to \infty$
- Zero points of classical potential function U

Plan of talk*

1. Results for Schrödinger operator $-\Delta + \lambda U(\cdot/\lambda)$
2. Definition of $P(\phi)_2$-Hamiltonian
3. Main Result 1 : $\lim_{\lambda\to\infty} E_1(\lambda)$
4. Main Result 2 :

$$\limsup_{\lambda\to\infty} \frac{\log\left(E_2(\lambda) - E_1(\lambda)\right)}{\lambda} \le -d_U^{Ag}(-h_0, h_0)$$

5. Properties of Agmon distance d_U^{Ag} and instanton (Existence of minimal geodesic and instanton, etc)

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Results for Schrödinger operators on $\mathbb{R}^N$

Assume

- $U \in C^\infty(\mathbb{R}^N)$, $U(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\liminf_{|x|\to\infty} U(x) > 0$.
- $\{x \mid U(x) = 0\} = \{x_1, \ldots, x_n\}$.
- $Q_i = \frac{1}{2} D^2 U(x_i) > 0$ for all i.

Then the first eigenvalue $E_1(\lambda)$ of $-\Delta + \lambda U(\cdot/\sqrt{\lambda})$ is simple and

$$\lim_{\lambda\to\infty} E_1(\lambda) = \min_{1\leq i\leq n} \mathrm{tr}\sqrt{Q_i}.$$

Tunneling for Schrödinger operators

In addition to the assumptions above, we assume the symmetry of U:

- $U(x) = U(-x)$,
- $\{x \mid U(x) = 0\} = \{-x_0, x_0\} \quad (x_0 \neq 0)$.

Let $E_2(\lambda)$ be the second eigenvalue. Then we have (due to Harrell, Jona-Lasinio, Martinelli and Scoppola, Simon, Helffer and Sjöstrand,...)

$$\lim_{\lambda\to\infty} \frac{\log(E_2(\lambda) - E_1(\lambda))}{\lambda} = -d_U^{Ag}(-x_0, x_0),$$

where $d_U^{Ag}(-x_0, x_0)$ is the Agmon distance between $-x_0$ and x_0:

Tunneling for Schrödinger operators

$$d_U^{Ag}(-x_0, x_0) = \inf\left\{ \int_{-T}^{T} \sqrt{U(x(t))}|\dot{x}(t)|dt \;\middle|\; \right.$$

x is a smooth curve on $\mathbb{R}^N$

$$\left. \text{with } x(-T) = -x_0,\, x(T) = x_0 \right\}.$$

Carmona and Simon (1981) gave another representation d_U^{CS} of d_U^{Ag} using an action integral:

$$d_U^{CS}(-x_0, x_0) = \inf\left\{ \int_{-\infty}^{\infty} \left(\frac{1}{4}|x'(t)|^2 + U(x(t)) \right) dt \right.$$

$$\left. \;\middle|\; \lim_{t\to-\infty} x(t) = -x_0,\; \lim_{t\to\infty} x(t) = x_0 \right\}.$$

Instanton

The minimizing path $x_E = x_E(t)$ $(-\infty < t < \infty)$ is called an instanton. The instanton x_E satisfies

$$x''(t) = 2(\nabla U)(x(t)).$$

Remark

The classical Newton's equation corresponding to $-\Delta + U$ *is* $x''(t) = -2(\nabla U)(x(t))$.

Instanton

Since $U(\pm x_0) = 0$, we have

$$d_U^{CS}(-x_0, x_0) = \inf\left\{\int_{-T}^{T}\left(\frac{1}{4}|x'(t)|^2 + U(x(t))\right)dt \;\middle|\; x(-T) = -x_0,\ x(T) = x_0,\ T > 0\right\}. \qquad (*)$$

Hence, by an elementary inequality $ab \le \frac{a^2+b^2}{2}$,

$$d_U^{Ag}(-x_0, x_0) \le d_U^{CS}(-x_0, x_0).$$

- Simon used $(*)$, Feynman-Kac formula and large deviation to prove tunneling estimate.

Free Hamiltonian

Let $m > 0$. Let μ be the Gaussian measure on $\mathcal{S}'(\mathbb{R})$ such that

$$\int_W {}_{\mathcal{S}(\mathbb{R})}\langle \varphi, w\rangle^2_{\mathcal{S}'(\mathbb{R})} d\mu(w) = \left((m^2 - \Delta)^{-1/2}\varphi, \varphi\right)_{L^2}.$$

Let $\mathcal{E}$ be the Dirichlet form defined by

$$\mathcal{E}(f, f) = \int_W \|\nabla f(w)\|^2_{L^2(\mathbb{R},dx)} d\mu(w) \quad f \in \mathrm{D}(\mathcal{E}),$$

where $\nabla f(w)$ is the unique element in $L^2(\mathbb{R}, dx)$ such that $\lim_{\varepsilon\to 0} \dfrac{f(w + \varepsilon\varphi) - f(w)}{\varepsilon} = (\nabla f(w), \varphi)_{L^2(\mathbb{R},dx)}$.

The generator $-L(\geq 0)$ of $\mathcal{E}$ is the free Hamiltonian.

Potential function of corresponding classical equation

Let $P(x) = \sum_{k=0}^{2M} a_k x^k$ with $a_{2M} > 0$.

Let $g \in C_0^\infty(\mathbb{R})$ with $g(x) \geq 0$ for all x and define for $h \in H^1 (= H^1(\mathbb{R}))$,

$$
\begin{aligned}
V(h) &= \int_{\mathbb{R}} P(h(x))g(x)dx \\
U(h) &= \frac{1}{4}\int_{\mathbb{R}} \left(h'(x)^2 + m^2 h(x)^2\right) dx + V(h)
\end{aligned}
$$

Wick product

We want to consider an operator like

$$-L + \lambda V(w/\sqrt{\lambda}) \quad \text{on} \quad L^2(\mathcal{S}'(\mathbb{R}), d\mu).$$

- Difficulty: w is an element of Schwartz distribution and $w(x)^k$ is meaningless.
- Renormalization is necessary:
  Wick product $\; : w(x)^k :$.

Potential function of $P(\phi)_2$ Hamiltonian

For $P = P(x) = \sum_{k=0}^{2M} a_k x^k$ with $a_{2M} > 0$, define

$$
\begin{aligned}
& \int_{\mathbb{R}} : P\left(\frac{w(x)}{\sqrt{\lambda}}\right) : g(x)dx \\
&= \sum_{k=0}^{2M} a_k \int_{\mathbb{R}} : \left(\frac{w(x)}{\sqrt{\lambda}}\right)^k : g(x)dx.
\end{aligned}
$$

We write

$$
\begin{aligned}
: V\left(\frac{w}{\sqrt{\lambda}}\right) : &= \int_{\mathbb{R}} : P\left(\frac{w(x)}{\sqrt{\lambda}}\right) : g(x)dx \\
V_\lambda(w) &= \lambda : V\left(\frac{w}{\sqrt{\lambda}}\right) : .
\end{aligned}
$$

Definition of Spatially cut-off $P(\phi)_2$-Hamiltonian

- $-L + V_\lambda$ is defined to be the unique self-adjoint extension operator of $(-L + V_\lambda, \mathfrak{F}C_b^\infty(\mathcal{S}'(\mathbb{R})))$.

- $-L + V_\lambda$ is bounded from below and the first eigenvalue $E_1(\lambda)$ is simple.
  The corresponding positive eigenfunction $\Omega_{1,\lambda}$ exists.

Main result 1

Assumption

(A1) $U(h) \geq 0$ *for all* $h \in H^1$ *and*

$$\mathcal{Z} = \{h \in H^1 \mid U(h) = 0\} = \{h_1, \ldots, h_n\}$$

is a finite set.

(A2) *The Hessian* $\nabla^2 U(h_i)$ $(1 \leq i \leq n)$ *is strictly positive.*

Remark

Since for any $h \in H^1$,

$$\nabla^2 U(h_i)(h,h) = \frac{1}{2}\int_{\mathbb{R}} h'(x)^2 dx + \int_{\mathbb{R}} \left(\frac{m^2}{2}h(x)^2 + P''(h_i(x))g(x)h(x)^2\right) dx,$$

the non-degeneracy is equivalent to

$$\inf \sigma(m^2 - \Delta + 4v_i) > 0,$$

where $v_i(x) = \frac{1}{2}P''(h_i(x))g(x)$.

Main Theorem 1

Theorem

Assume **(A1)** *and* **(A2)** *and let* $E_1(\lambda) = \inf \sigma(-L + V_\lambda)$. *Then*

$$\lim_{\lambda\to\infty} E_1(\lambda) = \min_{1\le i\le n} E_i,$$

where

$$\begin{aligned} E_i &= \inf \sigma(-L + Q_{v_i}), \\ Q_{v_i} &= \int_{\mathbb{R}} :w(x)^2: v_i(x)dx, \\ v_i(x) &= \frac{1}{2}P''(h_i(x))g(x). \end{aligned}$$

Cameron-Martin subspace of μ

Let $H^s(\mathbb{R})$ be the Sobolev space with the norm:

$$\|\varphi\|_{H^s(\mathbb{R})} = \|(m^2 - \Delta)^{s/2}\varphi\|_{L^2(\mathbb{R},dx)}.$$

Let $H = H^{1/2}(\mathbb{R})$. Then H is the Cameron-Martin subspace of μ and μ exists on $W \subset \mathcal{S}'(\mathbb{R})$:

$$W = \Big\{ w \in \mathcal{S}'(\mathbb{R}) \mid \|w\|_W^2 = \int_{\mathbb{R}} |(1 + |x|^2 - \Delta)^{-1} w(x)|^2 dx < \infty \Big\}.$$

- The triple (W, H, μ) is an abstract Wiener space.

Proof of the first main theorem

- IMS localization argument
- Lower bound estimate on the bottom of the spectrum of $-L + V_\lambda$ which follows from logarithmic Sobolev ineqaulities
- Large deviation principle and Laplace method for Wick polynomials (Wiener chaos)
- Gagliard-Nirenberg type estimate:

$$\left\{\int_{\mathbb{R}} |h(x)|^p g(x) dx\right\}^{1/p} \le C\|h\|_{H^{1/2}}^{a(s)} \|h\|_W^{1-a(s)},$$

where $a(s) = 3/(4-2s)$ and $\frac{p-2}{2p} < s < \frac{1}{2}$.

Tunneling for $P(\phi)_2$-Hamiltonians

Let

$$E_2(\lambda) = \inf\{\sigma(-L + V_\lambda) \setminus \{E_1(\lambda)\}\}.$$

It is known that $E_2(\lambda) > E_1(\lambda)$.

We prove that $E_2(\lambda) - E_1(\lambda)$ is exponentially small when $\lambda \to \infty$ in the case where the potential function is double well type.

Second main theorem

Assumption

(A3) *For all* x, $P(x) = P(-x)$ *and* $\mathcal{Z} = \{h_0, -h_0\}$, *where* $h_0 \neq 0$.

Theorem

Assume **(A1), (A2), (A3)**. *Then*

$$\limsup_{\lambda\to\infty} \frac{\log\left(E_2(\lambda) - E_1(\lambda)\right)}{\lambda} \leq -d_U^{Ag}(h_0, -h_0).$$

Example

Fix $g \in C_0^\infty(\mathbb{R})$.

For sufficiently large $a > 0$, the polynomial

$$P(x) = a(x^2 - 1)^{2n} - C$$

satisfies **(A1), (A2), (A3)**.

C is a positive constant which depends on a, g.

We define $d_U^{Ag}(-h_0, h_0)$.

Assumption

In the definition below, we always assume $U(h) \geq 0$ for all h.

Agmon distance on $H^1(\mathbb{R})$

Note that $h_0, -h_0 \in H^1(\mathbb{R})$.

Let $0 < T < \infty$ and $h, k \in H^1(\mathbb{R})$.

Let $AC_{T,h,k}(H^1(\mathbb{R}))$ be the all absolutely continuous paths $c : [0, T] \to H^1(\mathbb{R})$ satisfying $c(0) = h, c(T) = k$.

We define the Agmon distance between h, k by

$$d_U^{Ag}(h, k) = \inf \left\{ \ell_U(c) \mid c \in AC_{T,h,k}(H^1(\mathbb{R})) \right\},$$

where

$$\ell_U(c) = \int_0^T \sqrt{U(c(t))} \|c'(t)\|_{L^2} dt.$$

Agmon distance on $H^{1/2}(\mathbb{R})$

Agmon metric is conformal to L^2-metric. However the function U is defined on H^1. On which space the Agmon distance is naturally defined ?

- For any $h, k \in H^{1/2}(\mathbb{R})$, there exists $u(= u(t,x)) \in H^1((0,T) \times \mathbb{R})$ such that
  - $u(0,x) = h(x)$ and $u(T,x) = k(x)$,
  - $\int_0^T \sqrt{U(u(t))}\|u'(t)\|_{L^2}^2 dt < \infty$
- $H^1((0,T) \times \mathbb{R} \mid u(0) = h, u(T) = k) \subset \mathcal{P}_{T,h,k,U}$ which is defined in the next slide.

Agmon distance on $H^{1/2}(\mathbb{R})$

(1) Let $h, k \in H^{1/2}$. Let $\mathcal{P}_{T,h,k,U}$ be all continuous paths $c = c(t)$ $(0 \le t \le T)$ on $H^{1/2}$ such that

- $c(0) = h, c(T) = k$,
- $c \in AC_{T,h,k}(L^2(\mathbb{R}))$,
- $c(t) \in H^1(\mathbb{R})$ for $\|c'(t)\|dt$ -a.e. $t \in [0, T]$ and the length of c is finite:

$$\ell_U(c) = \int_0^T \sqrt{U(c(t))}\|c'(t)\|_{L^2} dt < \infty.$$

(2) Let $0 < T < \infty$. We define the Agmon distance between $h, k \in H^{1/2}(\mathbb{R})$ by

$$d_U^{Ag}(h, k) \ = \ \inf \{\ell_U(c) \mid c \in \mathcal{P}_{T,h,k,U}\}.$$

Proof in the case of Schrödinger operators

Assume

- $U \in C^\infty(\mathbb{R}^N)$, $U(x) \geq 0$ and $\liminf_{|x|\to\infty} U(x) > 0$,
- $U(x) = U(-x)$,
- $\{x \mid U(x) = 0\} = \{-x_0, x_0\} \quad (x_0 \neq 0)$,
- $\frac{1}{2} D^2 U(x_0) > 0$.

Then for the ground state $\Psi_{1,\lambda}$ of $-\Delta + \lambda^2 U$,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \Psi_{1,\lambda}(x) = -\min\left(d_U^{Ag}(x, x_0), d_U^{Ag}(x, -x_0)\right).$$

This and estimate on the second eigenfunction implies

$$\lim_{\lambda\to\infty} \frac{\log\left(E_2(\lambda) - E_1(\lambda)\right)}{\lambda} = -d_U^{Ag}(x_0, -x_0).$$

I-function of ground state measure for $P(\phi)_2$-Hamiltonians

Assume **(A1), (A2), (A3)**. Let

$$d\mu_{\lambda,U} = \Omega_{1,\lambda}^2 d\mu, \quad \mu_U^\lambda = (S_\lambda)_* \mu_{\lambda,U},$$

where $S_\lambda w = \frac{w}{\sqrt{\lambda}}$. Formally $d\mu_U^\lambda(w) = \Psi_{1,\lambda}(w)^2 dw$,

where $\Psi_{1,\lambda}$ is the ground state for

$$-\Delta_{L^2(\mathbb{R})} + \lambda^2 U(w) - \frac{\lambda}{2}\mathrm{tr}(m^2 - \Delta)^{1/2}.$$

It is natural to conjecture that μ_U^λ satisfies the large deviation principle with good rate function I_U:

$$I_U(h) = 2\min\left(d_U^{Ag}(h_0, h), d_U^{Ag}(-h_0, h)\right).$$

Approximation of Agmon distance

Assume U satisfies **(A1), (A2)**. Let $\mathcal{F}_U^W$ be the set of non-negative bounded globally Lipschitz continuous functions u on W which satisfy the following conditions.
(1) It holds that $0 \le u(h) \le U(h)$ for all $h \in H^1$ and

$$\{h \in H^1 \mid U(h) - u(h) = 0\} = \{h_1, \ldots, h_n\} = \{U = 0\}.$$

(2) u is C^2 in $\cup_{i=1}^n B_{\delta_0}(h_i)$ for some $\delta_0 > 0$ and

$$\inf \left\{ u(w) \,\middle|\, w \in \left(\cup_{i=1}^n B_\delta(h_i)\right)^c \right\} > 0 \quad \text{for any } \delta > 0,$$

where $B_\delta(h) = \{w \in W \mid \|w - h\|_W < \delta\}$.

Approximation of Agmon distance

(3) The Hessians

$$\nabla^2 (U - u)(h_i) \qquad (1 \le i \le n)$$

are strictly positive.

Approximation of Agmon distance

Let $\varphi, \psi \in L^2(\mathbb{R})$. Let $AC_{T,\varphi,\psi}\left(L^2(\mathbb{R})\right)$ be the set of all absolutely continuous paths $c : [0, T] \to L^2(\mathbb{R})$ with $c(0) = \varphi$ and $c(T) = \psi$. Let $u \in \mathcal{F}_U^W$. For $w_1, w_2 \in W$, define

- if $w_1 - w_2 \in L^2(\mathbb{R})$,

$$\rho_u^W(w_1, w_2) = \inf\Big\{ \int_0^T \sqrt{u(w_1 + c(t))} \|c'(t)\|_{L^2} dt \;\Big|\; c \in AC_{T,0,w_2-w_1}(L^2(\mathbb{R})) \Big\}.$$

- if $w_1 - w_2 \notin L^2(\mathbb{R})$, $\qquad \rho_u^W(w_1, w_2) = \infty.$

Approximation of Agmon distance

Lemma

Let $u \in \mathcal{F}_U^W$.

(1) *Let* O *be a non-empty open subset of* W *and set* $\rho_u^W(O, w) = \inf\{\rho_u^W(\phi, w) \mid \phi \in O\}$. *Then*

$$\rho_u^W(O, \cdot) \in \mathrm{D}(\mathcal{E}),$$

$$|\nabla \rho_u^W(O, w)|_{L^2(\mathbb{R}, dx)} \le \sqrt{u(w)} \quad \mu\text{-}a.s.w.$$

(2) *Assume* **(A1)**, **(A2)**. *Set* $u_\lambda(w) = \lambda u(w/\sqrt{\lambda})$, $E_1(\lambda, u) = \inf \sigma(-L_A + V_\lambda - u_\lambda)$. *Then* $\lim_{\lambda \to \infty} E_1(\lambda, u)$ *exists.*

Approximation of Agmon distance

Further define

$$\underline{\rho}_u^W(w_1, w_2) = \lim_{\varepsilon \to 0} \inf\left\{\rho_u^W(w, \eta) \,\middle|\, w \in B_\varepsilon(w_1), \eta \in B_\varepsilon(w_2)\right\}.$$

In the case where $W = H = \mathbb{R}^N$, for any w_1, w_2, clearly,

$$\sup_{u \in \mathcal{F}_U^W} \underline{\rho}_u^W(w_1, w_2) = d_U^{Ag}(w_1, w_2).$$

Approximation of Agmon distance

Lemma

Assume **(A1), (A2)**. *Then for all* $h, k \in H^{1/2}(\mathbb{R})$,

$$d_U^{Ag}(h,k) = \sup_{u \in \mathcal{F}_U^W} \rho_{-u}^W(h,k).$$

Exponential decay estimate for the ground state measure of $P(\phi)_2$-Hamiltonian

Lemma

Assume **(A1), (A2)**. *Let* $d\mu_{\lambda,U}(w) = \Omega_{1,\lambda}^2(w)d\mu$, *where* $\Omega_{1,\lambda}$ *is the ground state of* $-L + V_\lambda$. *Let* $r > \kappa$ *and* $0 < q < 1$. *Let* $B_\varepsilon(\mathcal{Z}) = \cup_{i=1}^n B_\varepsilon(h_i)$. *For large* λ,

$$\mu_{\lambda,U}\left(\left\{w \in W \,\middle|\, \rho_u^W\left(\frac{w}{\sqrt{\lambda}}, B_\varepsilon(\mathcal{Z})\right) \geq r\right\}\right) \leq \frac{C_1 e^{-2q\lambda(r-\kappa)}\|u\|_\infty}{\kappa^2(\lambda(1-q^2)\varepsilon^2 - C_2)},$$

where C_i *are positive constants independent of* λ, r, κ.

Proof of second main theorem

$$E_2(\lambda) - E_1(\lambda)$$
$$= \inf\left\{ \frac{\int_W |\nabla f(w)|_{L^2}^2 d\mu_{\lambda,U}(w)}{\int_W f(w)^2 d\mu_{\lambda,U}(w)} \;\middle|\; f \in \mathrm{D}(\mathcal{E}) \cap L^\infty(W,\mu), f \not\equiv 0, f \perp 1 \text{ in } L^2(\mu_{\lambda,U}) \right\}.$$

Assume **(A1), (A2), (A3)**.

- $\mathcal{Z} = \{h_0, -h_0\}$.
- $\mu_{\lambda,U}$ concentrates on neighborhoods of $\pm\sqrt{\lambda}h_0$.

Proof of second main theorem

- Let f be the function such that

$$f(w) = \begin{cases} 1 & \text{for } w \text{ near } \sqrt{\lambda}h_0 \\ -1 & \text{for } w \text{ near } -\sqrt{\lambda}h_0. \end{cases}$$

  Then $f \perp 1$ approximately in $L^2(\mu_{\lambda,U})$.
- f can be constructed by using functions $\rho_u^W\left(w/\sqrt{\lambda}, B_\varepsilon(h_0)\right)$, $\rho_u^W\left(w/\sqrt{\lambda}, B_\varepsilon(-h_0)\right)$.
  Also $\|\nabla f\|_\infty < \infty$ and

$$\mathrm{supp}|\nabla f| \subset \left\{ w \;\middle|\; \rho_u^W\left(\frac{w}{\sqrt{\lambda}}, B_\varepsilon(\mathcal{Z})\right) \approx \frac{\rho_{\underline{u}}^W(h_0, -h_0)}{2} \right\}$$

Properties of Agmon distance and instanton

- d_U^{Ag} can be extended to a continuous distance function on $H^{1/2}(\mathbb{R})$.
- The topology on $H^{1/2}(\mathbb{R})$ defined by d_U^{Ag} coincides with that of $(H^{1/2}(\mathbb{R}), \|\ \|_{H^{1/2}})$.
- Let $P = 0$. For any $h \in H^{1/2} \setminus H^1$ and $k(\neq 0) \in H^{1/2}$, $\limsup_{\varepsilon \to 0} \dfrac{d_U^{Ag}(h, h + \varepsilon k)}{\varepsilon} = +\infty.$
- Existence of minimal geodesic between h_0 and $-h_0$ (unique or not?).
- Existence of instanton.

Existence of minimal geodesic

Theorem

Assume **(A1), (A2)** *and $\mathcal{Z}$ consists of two points $\{h, k\}$. There exists a continuous curve $c_\star$ on $H^{1/2}(\mathbb{R})$ such that* $d_U^{Ag}(h,k) = \int_0^1 \sqrt{U(c_\star(t))}\|c_\star'(t)\|_{L^2}dt$ *and $c_\star$ satisifies the following.*

(1) $c_\star(0) = h$, $c_\star(1) = k$ *and* $c_\star(t) \neq h, k$ *for* $0 < t < 1$.

(2) $c_\star = c_\star(t,x)$ *is a* C^∞ *function of* $(t,x) \in (0,1) \times \mathbb{R}$ *and* $c_\star \in H^1((\varepsilon, 1-\varepsilon) \times \mathbb{R})$ *for all* $0 < \varepsilon < 1$.

(3) $\int_0^\varepsilon \|c_\star'(t)\|_{L^2}^2 dt = \int_{1-\varepsilon}^1 \|c_\star'(t)\|_{L^2}^2 dt = +\infty \ \forall \varepsilon > 0$.

Existence of instanton

$$\frac{\partial^2 u}{\partial t^2}(t,x) = 2(\nabla U)(u(t,x)) \tag{1}$$

The equation (1) reads

$$\frac{\partial^2 u}{\partial t^2}(t,x) + \frac{\partial^2 u}{\partial x^2}(t,x) = m^2 u(t,x) + 2P'(u(t,x))g(x) \tag{2}$$

Existence of instanton

Let $T > 0$ and define

$$I_{T,P}(u) = \frac{1}{4}\iint_{(-T,T)\times\mathbb{R}}\left(\left|\frac{\partial u}{\partial t}(t,x)\right|^2 + \left|\frac{\partial u}{\partial x}(t,x)\right|^2\right)dtdx + \iint_{(-T,T)\times\mathbb{R}}\left(\frac{m^2}{4}u(t,x)^2 + P(u(t,x))g(x)\right)dtdx$$

and

$$I_{\infty,P}(u) = \frac{1}{4}\int_{-\infty}^{\infty}\|\partial_t u(t)\|_{L^2(\mathbb{R})}^2 dt + \int_{-\infty}^{\infty} U(u(t))dt.$$

There exists a solution $u_\star = u_\star(t,x)$ $((t,x) \in \mathbb{R}^2)$ to the equation (2) which satisfies the following properties.

Existence of instanton

(1) It holds that
$u_\star|_{(-T,T)\times\mathbb{R}} \in H^1((-T,T)\times\mathbb{R}) \cap C^\infty((-T,T)\times\mathbb{R})$ for any $T>0$ and

$$I_{T,P}(u_\star|_{(-T,T)\times\mathbb{R}}) = \inf\{I_{T,P}(u) \mid u \in H^1_{T,u_\star(-T),u_\star(T)}(\mathbb{R})\},$$

where

$$H^1_{T,\varphi,\psi}(\mathbb{R}) = H^1((-T,T)\times\mathbb{R} \mid u(-T)=\varphi, u(T)=\psi).$$

Also we have

$$\lim_{t\to-\infty} \|u_\star(t) - h\|_{H^{1/2}} = 0$$
$$\lim_{t\to\infty} \|u_\star(t) - k\|_{H^{1/2}} = 0.$$

Existence of instanton

(2) It holds that $I_{\infty,P}(u_\star) = d_U^{Ag}(h,k)$ and $u_\star$ is a minimizer of the functional $I_{\infty,P}$ in the set of functions u satisfying the following conditions:

- $u|_{(-T,T)\times\mathbb{R}} \in H^1((-T,T),\mathbb{R})$ for all $T > 0$,
- $\lim_{t\to-\infty} \|u(t) - h\|_{H^{1/2}} = 0,$

  &

  $\lim_{t\to\infty} \|u(t) - k\|_{H^{1/2}} = 0.$

(3) Let $\mathcal{I}(T) = \inf \left\{ I_{T,P}(u) \mid u \in H^1_{T,h,k}(\mathbb{R}) \right\}$. Then $T \mapsto \mathcal{I}(T)$ is a strictly decreasing function and $\lim_{T\to\infty} \mathcal{I}(T) = d_U^{Ag}(h,k)$.

Relation between $c_\star$ and $u_\star$

Let

$$\rho(t) = \frac{1}{2d_U^{Ag}(h,k)} \int_{1/2}^{t} \|c'(s)\|_{L^2}^2 ds \quad 0 < t < 1,$$

$$\sigma(t) = \frac{1}{2d_U^{Ag}(h,k)} \int_{-\infty}^{t} \|u'(s)\|_{L^2}^2 ds \quad t \in \mathbb{R}.$$

Then $\rho^{-1}(t) = \sigma(t)$ $(t \in \mathbb{R})$ and

$$u_\star(t,x) = c_\star(\sigma(t),x) \quad t \in \mathbb{R},$$
$$u(\rho(t),x) = c(t,x) \quad 0 < t < 1.$$

Problems

- Lower bound estimate. WKB approximation. Precise estimate of $E_2(\lambda) - E_1(\lambda)$:

$$E_2(\lambda) - E_1(\lambda) = \exp(-\lambda d_U^{Ag}(h_0, -h_0)) \sum_{j=0}^{\infty} a_j \left(\frac{1}{\sqrt{\lambda}}\right)^{2j+1}$$

- Geometry of Agmon distance
- Relation between
  - Scattering of spatially cut-off $P(\phi)_2$-Hamiltonian (or $P(\phi)_2$-field itself after taking ∞-volume limit)
  - Scattering of non-linear Klein-Gordon equation (classical equation)

Appendix : Proof of $d_U^{Ag}(h,k) = \sup_u \rho^W_{\underline{u}}(h,k)$

Take large L so that $\mathrm{supp}\, g \subset [-L/8, L/8]$.
Let $w_L = \chi_L \cdot w$, where $\chi_L(x) = 1$ for $|x| \le L/4$, $\mathrm{supp}\, \chi_L \subset I = (-L/2, L/2)$.
Let P_N be a finite dimensional projection operator on $H^s_D(I, dx)$. Define

$$
\begin{aligned}
&u_{R,\varepsilon,N,L,\delta}(w) \\
&\quad = \left(\varepsilon\varepsilon_0 \min\left(u_{\mathcal{Z}}(w), 2R^2\right) + (1-\varepsilon)^3 U(P_N w_L)\right)\psi(w) \\
&\qquad + \min\left(\varepsilon_0 u_{\mathcal{Z}}(w), 2R^2\right)(1-\psi(w)),
\end{aligned}
$$

$u_{\mathcal{Z}}(w) = \min_{i=1,2} \|w - h_i\|_W^2$. Then

$$
u_{R,\varepsilon,N,L,\delta} \in \mathcal{F}_U^W, \quad \sup_{R,N,L,\delta,\varepsilon} \rho^W_{\underline{u}_{R,\varepsilon,N,L,\delta}}(h,k) = d_U^{Ag}(h,k).
$$

Appendix: Proof of $d_U^{Ag}(h,k) = \sup_u \underline{\rho}_u^W(h,k)$

ψ is defined as follows:

$$W_{R,N,L,\delta} = \Big\{ w \in W \ \Big|\ \|P_N w_L\|_{H_D^{1/2}(I)} \le R,$$
$$\|P_N^{\perp} w_L\|_{H_D^{-2}(I)} \le R, \min_{i=1,2} \|P_N w_L - h_i\|_{H^1(\mathbb{R})} \ge \delta \Big\},$$

where $h_1 = h_0, h_2 = -h_0$.

$$\psi(w) = \frac{d_W(w, \overline{W^c_{R,N,L,\delta}})}{d_W(w, W_{R/2,N,L,2\delta}) + d_W(w, \overline{W^c_{R,N,L,\delta}})}.$$